Optimal Combinations of Control Strategies for Dynamics of Endemic Malaria Disease Transmission

Dereje Gutema Edossa a* , Alemu Geleta Wedajo a and Purnachandra Rao Koya a

a Department of Mathematics, Wollega University, Nekemte, Ethiopia.

Authors’ contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this study, a non-linear system of ordinary differential equation model that describe the dynamics of malaria disease transmission is derived and analyzed. Conditions are derived from the existence of disease-free and endemic equilibria. Basic reproduction number $R_0$ of the model is obtained, and we investigated that it is the threshold parameter between the extinction and persistence of the disease. If $R_0$ is less than unity, then the disease-free equilibrium point is both locally and globally asymptotically stable resulting in the disease removing out of the host populations. The disease can persist whenever $R_0$ is greater than unity. At $R_0$ is equal to unity, existence conditions are derived from the endemic equilibrium for both forward and backward bifurcations. Furthermore, optimal combinations of time dependent control measures are incorporated to the model, and we derived the necessary conditions of the optimal control using Pontryagin’s maximum principal theory. Numerical simulations were conducted using MATLAB software to confirm our analytical results. Our findings were that malaria disease may be controlled more with strict application of the combination of all control measures that is, the combination of prevention of drug resistance, insecticide treated net ITN, indoor residual spray IRS and active treatment than when the combination of three control measures are used.

Keywords: Malaria; disease-free equilibrium; endemic equilibrium; basic reproduction number; stable; optimal control.

*Corresponding author: Email: derejegutema8@gmail.com;
1 Introduction

Malaria is an old infectious parasitic disease and transmitted to human through the bites of infected female Anopheles’ mosquitoes [1]. “The burden of malaria disease affects the community socioeconomic in many ways. Some of these are fertility, population growth, saving and investment, worker productivity, absenteeism, premature mortality and medical costs” [2]. “In areas where malaria is highly endemic, young children bears a larger burden in terms of the disease morbidity and mortality and affects fetal development during early stage of pregnancy in women due to loss of immunity. Malaria is still treating a serious challenge to the global world population. According to 2020 world health organization (WHO) report, 241 million cases and 627 thousand deaths from malaria globally and the estimate number of children under 5 years of age deaths caused by malaria only in Africa is 80%” [3].

The most popular strategies of controlling malaria disease includes, the use of chemotherapy, intermittent preventive treatment for children and pregnant women (preventive doses of sulfadoxine pyrimethamine (IPT/ST)), and use of insecticides treated bed nets and insecticides against the vector. The challenge posed by the resistance of parasites against drugs and resistance of mosquitoes against insecticides calls for urgent need for a better understanding of important parameters in the disease transmission and develops effective and optimal strategies for prevention and control of the spread of malaria disease

“Mathematical modeling has become an important tool in understanding the complex dynamics of disease transmission and in decision making processes regarding intervention programs for disease control. Concerning malaria disease, Ross (1911) developed the first mathematical model. He focused his study on mosquito control and showed that for the disease to be eliminated the mosquito population should be brought below a certain threshold” [4]. Later the idea of Ross is extended by Macdonald to account for super infection [5]. Ngwa, G. A. Shu, W.S., A mathematical model for endemic malaria with variable human and mosquito population [6]. Alemu G. W., Boka K.B., P.R. Koya derived and analyzed deterministic model for the inclusion of Infected immigrants on the spread and dynamics of malaria transmission [7], Chiyaka,C., Garira, and W., Dube, S., derived analyzed effects of treatment and drug resistance on the transmission dynamics of malaria in endemic areas [8], J.Tumwiine, S.D.H-Musekwa and F.Nyabadaza were analyzed a mathematical model for the transmission and spread of drug sensitive and resistant malaria strains within human populations [9]. Other studies are carried out by using optimal control theory. Okosun et al. derived and analyzed a malaria disease transmission mathematical model that includes treatment and vaccination with waning immunity and applied optimal control to study the impact of a possible vaccination with treatment strategies in controlling the spread of malaria [10], F.B. Agusto, and M.A. Khan, derived and analyzed Optimal Control Strategies for dengue transmission [11], K. O. Okosun and O. D. Makinde Modelling the impact of drug resistance in malaria transmission and its optimal control analysis [12], E. Bonyah, M.A. Khan,K.O. Okosun, J.F. Gómez-Aguilar present ”Modeling the effects of heavy alcohol consumption on the transmission dynamics of gonorrhea with optimal [13]. Makinde and Okuson, were applied optimal control to study the impact of chemo-therapy on malaria disease with infective immigrants [14], K. O. Okuson, O. Rachid, and N. Marcus, applied optimal control strategies and cost-effectiveness analysis of a malaria model [15]. Temesgen D. K., O. D.Makinde & Legesse L. O. derived and analyzed Optimal Control and Cost Effectiveness Analysis of SIRS Malaria Disease Model with Temperature Variability[16].

In this paper, we study SITRS-SI and SIRS-SI endemic malaria transmission model with standard incidence law that was presented by [12]. Furthermore, we modified the model [6] by omitting the incubating class from the system and incorporate four time dependent control measures, the class infective in treatment individuals and infectious classes with drug sensitive and drug resistant individuals. The purpose of this study is

(i) to investigate the stability for both disease-free equilibrium and endemic equilibrium
(ii) to develop effective ways for controlling the malaria disease
(iii) to explore the best strategy in terms of reducing the number of malaria infectious populations to zero.

2 Model Description and Formulation

The model subdivides the human populations in to five sub class namely, susceptible $S_r$, infected with drug sensitive malaria strain $I_{hss}$, infected with drug resistant malaria strain $I_{hrs}$, infective in treatment $T_r$, recovered
Similarly, the mosquito populations are also subdivided into susceptible class $S_v$, and infected class $I_v$. The total number of human and mosquito populations at time $t$ are denoted and given by $N_h(t) = S_h(t) + I_{hs}(t) + I_{hr}(t) + R_h(t)$ and $N_v(t) = S_v(t) + I_v(t)$ respectively. Note that, $S_h = S_h(t)$, $I_{hs} = I_{hs}(t)$, $I_{hr} = I_{hr}(t)$, $T_h = T_h(t)$, $R_h = R_h(t)$, $S_v = S_v(t)$, $I_v = I_v(t)$ and $N_v = N_v(t)$. The susceptible humans $S_h$ are recruited at the rate of $\Lambda_v$ and they either die from natural causes at a rate of $\mu_h$ or move to infected class $I_h$ by acquiring malaria through contact with infectious mosquitoes with respective rate of force of infection $\lambda_h = \phi_h \beta_h \frac{S_v}{N_v}$. Where, $\beta_h$ is the rate of probability of human getting infected, $\phi$ is the mosquito contact rate with human and $\omega$ is mosquito biting rate. We also let a fraction $\rho$ of humans be infected with drug sensitive malaria strain and the remaining fraction $(1 - \rho)$ individuals are infected with drug resistant malaria strains. Infected humans with drug sensitive malaria strains $I_{hs}$ individuals are either die from natural causes and due to disease death with respective rates $\mu_h$ and $\delta_h$ respectively or move to infective in treatment $T_h$ at a rate $\alpha$ and recovered class $R_h$ at recovery rate $\gamma_h$. Infected with drug resistant malaria strains $I_{hr}$ individuals are also either die from natural causes and due to disease death with respective rates $\mu_h$ and $\delta_h$ respectively or move to infective in treatment $T_h$ at a rate $\sigma$ and recovered class $R_h$ at recovery rate $\gamma_r$. Infective in treatment $T_h$ individuals are individuals with malaria disease that are getting treated under the control. They also either die from natural causes and due to disease death with respective rates $\mu_h$ and $\delta_h$ respectively or move to the susceptible class with fraction of $\epsilon$ due to the administered drug kills off the parasites and the infected humans with drug resistant malaria strains class with fraction of $\sigma$ due to treatment failure. These infected with drug sensitive and resistant malaria strains individuals progress to partially immune group (recovered class). Partially immune group (recovered individuals) either loses immunity and becomes again move to susceptible class with respective rate $\theta$ or die from natural death at a rate $\mu_v$. Susceptible mosquitoes $S_v$ are recruited at the rate $\Lambda_v$. They either die due to natural death at a rate of $\mu_v$ or move to infected class $I_v$ by acquiring malaria through contact with infectious humans with respective rate of force of infection $\lambda_v = \phi_v \beta_v \left( \frac{I_{hs} + I_{hr}}{N_h} \right)$. Where $\beta_v$ is the probability of a mosquito getting infected. Infected mosquitoes $I_v$ are die because of natural and disease induced death with respective rates $\mu_v$ and $\delta_v$ respectively. No recovered compartment for mosquitoes. We represent diagrammatically the flow of both the human and mosquito populations from one class to the other is given below.

Fig. 1. Flow diagram for the transmission of endemic malaria model

\[
\begin{align*}
\frac{dS_h}{dt} &= \Lambda_h + \varepsilon T_h + \theta R_h - (\lambda_h + \mu_h)S_h \\
\frac{dI_{hs}}{dt} &= \rho(1 - \pi)\lambda_h S_h - (\mu_h + \delta_h + \alpha + \gamma_s)I_{hs} \\
\frac{dI_{hr}}{dt} &= (1 - (1 - \pi)\rho)\lambda_h S_h - (\mu_h + \delta_h + \gamma_r + \sigma)I_{hr} \\
\frac{dT_h}{dt} &= \alpha I_{hs} + \sigma I_{hr} - (\delta_h + \mu_h + \varepsilon)T_h \\
\frac{dR_h}{dt} &= \gamma_s I_{hs} + \gamma_r I_{hr} - (\theta + \mu_h)R_h \\
\frac{dS_v}{dt} &= \Lambda_v - (\lambda_v + \mu_v)S_v \\
\frac{dI_v}{dt} &= \lambda_v S_v - (\mu_v + \delta_v)I_v
\end{align*}
\] (2.1)
With initial conditions
\[ S_h(0) = S_0, \quad I_h(0) = I_{0h}, \quad I_{hr}(0) = I_{0hr}, \quad T_h(0) = T_{0h}, \quad R_h(0) = R_{0h}, \quad S_v(0) = S_{0v} \]
\[ I_v(0) = I_{0v} \]  
(2.2)

With some of the following additional assumptions

(i) The susceptible class in both the human and mosquito populations enter into the infective classes by adequate contact with infectious populations not infective in treatment.
(ii) Infective in treatment and recovered individuals are not infectious to the susceptible populations
(iii) Those infective humans recovered from the disease due to natural immunity and enter into a partially immune group
(iv) Those infective individuals in treatment recovered from the disease due to the administered drug killed off the parasites
(v) One part of the recovered class again becomes susceptible to the disease
(vi) No recovered compartment for mosquitoes.

3 Basic Property of the Model

3.1 Positivity of the model

**Theorem 1** Every solution of system (2.1) with initial conditions equation (2.2) exists in the

Interval \([0, \infty)\) and \(S_h(t) \geq 0, I_h(t) \geq 0, I_{hr}(t) \geq 0, T_h(t) \geq 0, R_h(t) \geq 0, S_v(t) \geq 0\) and \(I_v(t) \geq 0\) for all \(t \geq 0\).

Proof. To show positivity of solutions, it is enough to show that each of the trajectories of system (2.1) is non-negative for all \(t \geq 0\).

Proof. To show positivity of solutions, it is enough to show that each of the trajectories of system (2.1) is non-negative for all \(t \geq 0\).

Since the right-hand side of system (2.1) is completely continuous and locally Lipschitzian on \(C\), the solution \((S_h(t), I_h(t), I_{hr}(t), T_h(t), R_h(t), S_v(t), I_v(t))\) of system (2.1) with initial condition equation (2.2) exists and unique on \([0, k)\) where \(0 < k < +\infty\).

It follows from the first system (2.1) that, the differential inequality describing the evolution of the susceptible human population over time \(t\) is given by
\[
\frac{dS_h}{dt} \geq \Lambda_h - \left( \phi \alpha_h \left( \frac{I_v}{N_h} \right) (t) + \mu_h \right) S_h(t)
\]
\[
\frac{d}{dt} \left[ S_h(t) \exp \left\{ \mu_h t + \int_0^t \phi \alpha_h \left( \frac{I_v}{N_h} \right) (s) \, ds \right\} \right] \geq \Lambda_h \exp \left\{ \mu_h t + \int_0^t \phi \alpha_h \left( \frac{I_v}{N_h} \right) (s) \, ds \right\}
\]
Hence,
\[
S_h(t) \exp \left\{ \mu_h t + \int_0^t \phi \alpha_h \left( \frac{I_v}{N_h} \right) (s) \, ds \right\} \geq S_0 \exp \left\{ \int_0^t \Lambda_h \exp \left\{ \mu_h t + \int_0^t \phi \alpha_h \left( \frac{I_v}{N_h} \right) (s) \, ds \right\} \, ds \right\} dt
\]

Thus,
\[
S_h(t) \geq S_0 \exp \left\{ - \mu_h t + \int_0^t \phi \alpha_h \left( \frac{I_v}{N_h} \right) (s) \, ds \right\} + \exp \left\{ - \mu_h t + \int_0^t \phi \alpha_h \left( \frac{I_v}{N_h} \right) (s) \, ds \right\} \right\} \times 
\int_0^t \Lambda_h \exp \left\{ \mu_h t + \int_0^t \phi \alpha_h \left( \frac{I_v}{N_h} \right) (s) \, ds \right\} \, ds \right\} dt > 0.
\]
From the second system (2.1) we have,
\[ \frac{dh_x}{dt} \geq -(\mu_h + \delta_h + \alpha + \gamma_x)h_x(t) \text{ is equivalent to } h_x(t) \geq \exp[-(\mu_h + \delta_h + \alpha + \gamma_x)t] > 0. \]

From the third system (2.1) we have,
\[ \frac{dh_r}{dt} \geq -(\mu_h + \delta_h + \gamma_r + \sigma)h_r(t) \text{ is equivalent to } h_r(t) \geq \exp[-(\mu_h + \delta_h + \gamma_r + \sigma)t] > 0. \]

From the fourth system (2.1) we have,
\[ \frac{dr_h}{dt} \geq -(\delta_h + \mu_h + \epsilon)h_r(t) \text{ is equivalent to } h_r(t) \geq \exp[-(\mu_h + \epsilon)t] > 0. \]

From the fifth system (2.1) we have,
\[ \frac{dr_h}{dt} \geq -(\mu_h + \theta)h_r(t) \text{ is equivalent to } h_r(t) \geq \exp[-(\mu_h + \theta)t] > 0. \]

From the sixth system (2.1) we have,
\[
\frac{dS_v}{dt} \geq \Lambda_v - \left( \int_0^t \phi \omega \beta_v \left( \frac{I_{hx} + I_{hr}}{N_h} \right)(s) ds \right) \geq \Lambda_v \exp \left\{ \mu_v t + \int_0^t \phi \omega \beta_v \left( \frac{I_{hx} + I_{hr}}{N_h} \right)(s) ds \right\}.
\]

Hence,
\[
S_v(t) \geq S_v(0) \exp \left\{ -\left( \mu_v t + \int_0^t \phi \omega \beta_v \left( \frac{I_{hx} + I_{hr}}{N_h} \right)(s) ds \right) \right\} - S_v(t) \geq \int_0^t \Lambda_v \exp \left\{ \mu_v t + \int_0^t \phi \omega \beta_v \left( \frac{I_{hx} + I_{hr}}{N_h} \right)(s) ds \right\} dt.
\]

Thus,
\[
S_v(t) \geq S_v(0) \exp \left\{ -\left( \mu_v t + \int_0^t \phi \omega \beta_v \left( \frac{I_{hx} + I_{hr}}{N_h} \right)(s) ds \right) \right\} + \exp \left\{ -\left( \mu_v t + \int_0^t \phi \omega \beta_v \left( \frac{I_{hx} + I_{hr}}{N_h} \right)(s) ds \right) \right\} \times \int_0^t \Lambda_v \exp \left\{ \mu_v t + \int_0^t \phi \omega \beta_v \left( \frac{I_{hx} + I_{hr}}{N_h} \right)(\psi) d\psi \right\} dt > 0.
\]

From the seventh system (2.1) we have,
\[ \frac{di_v}{dt} \geq -(\mu_v + \delta_v)i_v(t) \text{ is equivalent to } i_v(t) \geq \exp[-(\mu_v + \delta_v)t] > 0. \]

Therefore, we can see that \( S_v(t) > 0 \), \( I_{hx}(t) > 0 \), \( I_{hr}(t) > 0 \), \( T_v(t) > 0 \), \( R_v(t) > 0 \), \( S_v(t) > 0 \), \( I_v(t) > 0 \) for all \( t \geq 0 \).

### 3.2 Invariant region

**Theorem 2** The feasible region \( \Gamma \) defined by

\[
\Omega = \{ \Omega_h \times \Omega_v \} \subset \{ \mathbb{R}_+^5 \times \mathbb{R}_+^2 \}, \text{ where, } \Omega_h = \left\{ (S_v, I_{hx}, I_{hr}, T_v, R_v) \in \mathbb{R}_+^5 : N_h \leq \frac{\Lambda_h}{\mu_h} \right\} \text{ and } \\
\Omega_v = \{ (S_v, I_v) \in \mathbb{R}_+^2 : N_v \leq \frac{\Lambda_v}{\mu_v} \}, \text{ with initial conditions } S_h(0) = S_v(0) = I_{hx}(0) = I_{hr}(0) = T_v(0) = R_v(0) = R_h(0) = S_v(0), I_v(0) = 0, \text{ is bounded.}
\]

**Proof:** Let \( N_v(t) = S_v(t) + I_{hx}(t) + I_{hr}(t) + T_v(t) + R_v(t) \) and \( N_v(t) = S_v(t) + I_v(t) \)
The feasible region of both the human and mosquito populations are determined by the feasible region of \( N_h(t) \) and \( N_v(t) \) respectively as follows

**The feasible region of \( N_h(t) \):** Total sum of human compartments of system (2.1) leads to

\[
\frac{dN_h}{dt} = \Lambda_h - \mu_h N_h(t) - \delta_h (I_h(t) + I_v(t) + T_h(t)) \quad \text{if and only if} \quad \frac{dN_h}{dt} \leq \Lambda_h - \mu_h N_h(t) \quad \text{if and only if} \quad \frac{dN_h}{dt} + \mu_h N_h(t) \leq \Lambda_h
\]

The resulting differential inequality can be solved by separation of variables to give,

\[
\int \frac{d}{dt} (N_h e^{\mu_h t}) \leq \int \Lambda_h e^{\mu_h t} \\
\int \frac{d}{dt} (N_h e^{\mu_h t}) \leq \int \Lambda_h e^{\mu_h t}
\]

Taking the initial conditions \( t = 0 \) and denoting \( N_h(0) \) by \( N_{0h} \), then the complete solution

\[
N_h(t) \leq \frac{\Lambda_h}{\mu_h} + \left( N_{0h} - \frac{\Lambda_h}{\mu_h} \right) e^{-\mu_h t}.
\]

As \( t \to \infty, 0 < N_h \leq \frac{\Lambda_h}{\mu_h} \). So if \( N_{0h} \leq \frac{\Lambda_h}{\mu_h} \), then \( \lim_{t \to \infty} N_h(t) \leq \frac{\Lambda_h}{\mu_h} \). This means that \( \frac{\Lambda_h}{\mu_h} \) is upper bound of \( N_h \). On the other hand if \( N_{0h} > \frac{\Lambda_h}{\mu_h} \), then \( N_h(t) \) will decrease to \( \frac{\Lambda_h}{\mu_h} \). Thus \( N_{0h} \leq N_h(t) \leq \frac{\Lambda_h}{\mu_h} \). Therefore; the total human population is bounded.

**The feasible region of \( N_v(t) \):** total sum of mosquito compartments of the system (2.1)

leads to

\[
\frac{dN_v}{dt} = \Lambda_v - \mu_v N_v - \delta_v I_v \\
\frac{dN_v}{dt} \leq \Lambda_v - \mu_v N_v(t) \\
\frac{dN_v}{dt} + \mu_v N_v(t) \leq \Lambda_v
\]

The resulting differential inequality can be solved by separation of variables to give,

\[
\int \frac{d}{dt} (N_v e^{\mu_v t}) \leq \int \Lambda_v e^{\mu_v t}
\]

Taking the initial conditions \( t = 0 \) and denoting \( N_v(0) \) by \( N_{0v} \), then the complete solution

\[
N_v(t) \leq \frac{\Lambda_v}{\mu_v} + \left( N_{0v} - \frac{\Lambda_v}{\mu_v} \right) \exp\left(-\mu_v t\right).
\]

As \( t \to \infty, 0 < N_v \leq \frac{\Lambda_v}{\mu_v} \). So if \( N_{0v} \leq \frac{\Lambda_v}{\mu_v} \), then \( \lim_{t \to \infty} N_v(t) \leq \frac{\Lambda_v}{\mu_v} \). This means that \( \frac{\Lambda_v}{\mu_v} \) is upper bound of \( N_v \). On the other hand if \( N_{0v} > \frac{\Lambda_v}{\mu_v} \), then \( N_v(t) \) will decrease to \( \frac{\Lambda_v}{\mu_v} \). Thus \( N_{0v} \leq N_v(t) \leq \frac{\Lambda_v}{\mu_v} \).

Therefore; the total mosquito population is bounded. Thus, the solutions of the model variables representing human populations \( \{(S_h, I_h, T_h, R_h)\} \)

are confined in the feasible region \( \Omega_h = \{(S_h(t), I_h(t), T_h(t), R_h(t)) \in \mathbb{R}_+^4 : N_h \leq \frac{\Lambda_h}{\mu_h} \} \). Similarly, the solutions of the model variables representing mosquito populations \( \{(S_v, I_v)\} \) are confined in the feasible region \( \Omega_v = \{(S_v, I_v) \in \mathbb{R}_+^2 : N_v \leq \frac{\Lambda_v}{\mu_v} \} \).
This shows that the feasible region of the system system (2.1) is bounded and is given by \( \Omega = \{ S_h(t), I_h(t), I_{hr}(t), T_h(t), R_h(t), S_v(t), I_v(t) \} \in \mathbb{R}_+^7 \) or equivalent to \( \Omega = \{ \Omega_h \times \Omega_v \} \subset \{ \mathbb{R}_+^5 \times \mathbb{R}_+^2 \} \).

Thus, in \( \Omega \) the system (2.1) is well-posed epidemiologically and mathematically. Hence, it is sufficient to study the dynamics of the model in \( \Omega \).

### 4 Disease-free Equilibrium and Basic Reproduction Number, Disease-free Stability

The disease-free equilibrium point of the model is its steady state solutions without infection or disease.

Consider the disease-free equilibrium point denoted and given by:

\[
E_0 = (S_h^0, I_h^0, I_{hr}^0, T_h^0, R_h^0, S_v^0, I_v^0)
\]

Where, \( S_h^0, I_h^0, I_{hr}^0, T_h^0, R_h^0, S_v^0 \) and \( I_v^0 \) are the components of \( E_0 \) and \( I_h^0 = I_{hr}^0 = T_h^0 = R_h^0 = I_v^0 = 0 \) and the non-infectious are obtained by setting \( \frac{dS_h}{dt} = \frac{dS_v}{dt} = 0 \) for the malaria model system (2.1) and after computing the resultant gives \( S_h^0 = \frac{\Delta_h}{\mu_h} \) and \( S_v^0 = \frac{\Delta_v}{\mu_v} \). Hence,

\[
E_0 = \left( \frac{\Delta_h}{\mu_h} 0 0 0 0 \frac{\Delta_v}{\mu_v} 0 \right)
\]  

(4.1)

The basic reproduction number denoted by \( R_0 \) is the average number of secondary infectious infected by an infectious individual during his or her whole course of disease [17]. We use the next generation matrix method by van den Driessche and Watmough [18] to derive the basic reproduction number \( R_0 \) of system (2.1). The infectious compartment of system (2.1) are, \( I_h, I_{hr}, \) and \( I_v \). To apply the method [18], let the system (2.1) be rearranged by beginning with the infected classes as follows:

Let \( X = \left( l_{hs} I_h T_h I_v S_h R_h S_v \right)^T \)

\[
F(X_i) = \begin{pmatrix}
0 & 0 & (1 - \pi) \rho \phi \omega \beta_h \\
0 & 0 & (1 - (1 - \pi)) \rho \phi \omega \beta_h
\end{pmatrix}
\]

and \( V(X_i) = \begin{pmatrix}
(\mu_h + \delta_h + \alpha + \gamma_s) l_{hs} \\
(\mu_v + \delta_v) l_v
\end{pmatrix} \)

The new infection matrix \( F \) and the transition matrix \( V \) are given, respectively, by

\[
F = \begin{pmatrix}
0 & 0 & (1 - \pi) \rho \phi \omega \beta_h & 0 & 0 \\
0 & 0 & (1 - (1 - \pi)) \rho \phi \omega \beta_h & 0 & 0
\end{pmatrix}
\]

and

\[
V = \begin{pmatrix}
(\mu_h + \delta_h + \alpha + \gamma_s) & 0 \\
0 & (\mu_v + \delta_v)
\end{pmatrix} \)

The basic reproduction number of system (2.1) is the dominant eigen value of the next generation matrix \( FV^{-1} \) which is given by
4.1 Local stability of disease-free equilibrium point

**Theorem 3** The disease-free equilibrium point $E_0$ of system (2.1) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

**Proof:**

The local stability of the system is determined by the signs of the eigenvalues and it is further proved by linearizing to obtain its Jacobian at disease-free steady-state points so that

The Jacobian matrix of system (2.1) at disease free equilibrium point $E_0$ defined and given by

$$J(E_0) = \begin{pmatrix} -\mu_h & 0 & 0 & \varepsilon & 0 & 0 & -m_{17} \\ 0 & -m_1 & 0 & 0 & 0 & 0 & m_{27} \\ 0 & 0 & -m_2 & 0 & 0 & 0 & m_{37} \\ 0 & \alpha & \sigma & -m_3 & 0 & 0 & 0 \\ 0 & \gamma_s & \gamma_r & 0 & -m_4 & 0 & 0 \\ 0 & -m_{62} & -m_{63} & 0 & 0 & -\mu_v & 0 \\ 0 & m_{72} & m_{73} & 0 & 0 & 0 & -m_5 \end{pmatrix}$$

Where, $m_1 = (\mu_h + \delta_h + \alpha + \gamma_s)$, $m_2 = (\mu_h + \delta_h + \sigma + \gamma_r)$, $m_3 = (\mu_h + \delta_h + \varepsilon)$, $m_4 = (\mu_h + \theta)$, $m_5 = (\mu_v + \delta_v)$, $m_{17} = \phi_0 \omega \beta_h$, $m_{27} = (1-\pi)\rho \phi_0 \beta_h$, $m_{37} = (1-(1-\pi)\rho) \phi_0 \beta_h$, $m_{62} = m_{63} = m_{72} = m_{73} = 0$.

$$\text{Det}(J(E_0) - \lambda I) = 0$$

If and only if $\lambda_1 = -\mu_h < 0$, $\lambda_2 = -\mu_v < 0$, $\lambda_3 = -m_2 < 0$,

$$\lambda_4 = -m_4 < 0$$

and

$$a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0$$

Where,

$$a_0 = 1,$$

$$a_1 = m_1 + m_2 + m_5,$$

$$a_2 = m_2 m_3 + m_1 (m_2 + m_5) - \frac{\phi^2 \omega^2 \beta_h \Delta w \mu_2}{\Delta \beta h \mu},$$

$$a_3 = m_1 m_2 m_5 (1 - R_0^2)$$

By the principle of Ruth-Hurwitz criteria [19], equation (4.3) has negative real eigenvalues if and only if $a_1 > 0, a_2 > 0$ and $a_3 > 0$. Clearly, we see that $a_1 > 0$ because of it is the sum of positive variables, but $a_3 > 0$ if and only if $1 - R_0^2 > 0$ which is equivalent to $R_0 < 0$ and hence, all eigenvalues of the determinant of equation (4.3) will have negative real eigenvalues. Therefore; the disease-free equilibrium point $E_0$ is locally asymptotically stable.

5 Existence of Endemic Equilibrium and Bifurcation, and Local Stability of Endemic Equilibrium

Let $E^* = (S^*_h, I^*_h, I^*_r, T^*_h, R^*_h, S^*_v, I^*_v)$ be a non-trivial endemic equilibrium point of system (2.1), that is all components of $E^*$, are positive. If we set system (2.1) to zero we get the following
Where, and substituting (5.1) in to (5.3) we get

\[ \lambda_0^* = \phi \omega \beta_h \frac{l_0^*}{N_h^*} \]  

Again substituting (5.1) and (5.4) respectively in to (5.2) we get

\[ \lambda_0^*(b_0(\Lambda_h^*)^2 + b_1 \Lambda_h^* + b_2) = 0 \]  

Where,

\[ b_0 = k_3 m_1 m_2 m_3 m_4 \Lambda_h \left( \mu_v k_3 + m_2 \phi \omega \beta_v \right) \]
\[ b_1 = \frac{m_1 m_2 m_3 m_4 k_4}{m_1 m_2 m_3 m_4 + k_4 \Lambda_h} \]
\[ b_2 = m_1^2 m_2^2 m_3^2 m_4 \Lambda_h (1 - R_0^2) \]  

Where, \( R_0 \) is the basic reproduction number given by (4.2) and

\[ R_0 = \sqrt{\frac{\mu_v m_1 \left( 1 - \pi (1 - \pi) \mu_v m_2 + \phi \omega \beta_v m_2 m_3 m_4 \left( (1 - \pi) \mu_v m_2 + (1 - (1 - \pi) \mu_v m_2) \right) \right)}{m_1^2 m_2^2 m_3^2 m_4 \left( \mu_v k_3 + m_2 \phi \omega \beta_v \right)}} \]

Equation (5.5) admits a trivial solution \( \lambda_0^* = 0 \) which corresponds to the disease-free equilibrium point (DFEP).

Now we assume \( \lambda_0^* \neq 0 \) the existence of endemic equilibria is regulated by the quadratic equation \( b_0(\Lambda_h^*)^2 + b_1 \Lambda_h^* + b_2 = 0 \). The coefficient \( b_0 \) in (5.6) is always positive and \( b_2 \) is positive if \( R_0 < 1 \) and negative if \( R_0 > 1 \). So, the sign of \( b_1 \) and \( b_2 \) will decide about the positive solution of (5.5). For the case when \( R_0 > 1 \), two solutions can be obtained for (5.5), that are positive and negative. For the case when considering \( b_2 = 0 \) if and only if \( R_0 = 1 \), then a solution of the form \( \lambda_0^* = -\frac{b_1}{b_2} \) exists when \( b_1 < 0 \) \(( R_0 < R_0^* ) \). It follows that the number of endemic equilibria of (2.1) depend on the coefficient \( b_0, b_1 \) and \( b_2 \) as follows:

**Theorem 4**

The system (1) has

(i) a unique endemic equilibrium if \( b_2 < 0 \) if and only if \( R_0 > 1 \)

(ii) a unique endemic equilibrium if \( b_1 < 0 \) and \( b_2 > 0 \) or \( b_1 < 0, b_2 > 0 \) and \( b_2^2 - 4b_1b_2 = 0 \)

(iii) Two endemic equilibrium if \( b_2 > 0 \) and \( b_1 < 0 \) and \( b_2^2 - 4b_1b_2 > 0 \)

(iv) otherwise no endemic equilibrium

Here also, when put for the value of \( \Lambda_h = 0.071 \) from [20] and use Table 1 for the values of other parameters, the two roots are presented graphically as shown in Fig. 2. Where, the blue line represents stable equilibrium and the red line represents unstable equilibrium.
Edossa et al.; JAMCS, 37(8): 11-33, 2022; Article no.JAMCS.91316

When we plot the basic reproduction number $R_0$ versus the force of infection mosquitoes to humans, we note stable disease free region when $\lambda^*_1 = 0$ and when $R_0 = 1$, the force of infection mosquitoes to humans starts to increase in stable endemic region where we note that the disease start to spread again and hence, forward bifurcation.

5.1 Existence of back ward bifurcation

To show the existence of backward bifurcation of system (2.1), we employ the method developed in Gumel and Song, 2008; Castillo-Chavez and Song, 2004 [21-23]. We also assume that, the normal form representing the dynamics of the system on the Centre manifold theory is given by $\hat{\mu} = a\mu^2 + b\xi\mu$, where,

$$a = \frac{v}{2}D_{xx} f(x_0, 0)w^2 = \frac{1}{2}\sum_{k,i,j=1}^{n}v_kw_iw_j\frac{\partial^2 f_k}{\partial x_i \partial x_j}(x_0, 0) \neq 0 \text{ for } j = 1, 2, \ldots, n$$

(5.7)

$$b = V.D_{xx} f(x_0, 0)w = \sum_{k,i=1}^{n}v_kw_i\frac{\partial^2 f_k}{\partial x_i \partial \xi}(x_0, 0) \neq 0 \text{ for } i = 1, 2, \ldots, n$$

(5.8)

Where,

$\xi$ Denotes a bifurcation parameter to be chosen,

$n$ Denote the right hand side of system (2.1),

$x$ Denotes the state vector,

$x_0$ Denotes the disease-free equilibrium $E_0$,

$D_x$ Denotes the differential operator with respect to $x$,

$D_\xi$ Denotes the differential operator with respect to $\xi$, and

$w_0, v_0$ denotes the right and left eigenvectors respectively corresponding to the null eigenvalue of the Jacobian matrix of (2.1), evaluated at $x_0$ for $\xi = 0$.

Let we choose the rate of transmission of infection from an infectious mosquito to a susceptible human $\beta_h$ as the bifurcation parameter. We observe that $R_0 = 1$ is equivalent to:

$$\beta_h = \beta_h^* = \frac{\lambda^*_1m^*_1m^*_2m^*_3}{\phi^2\omega^2(\mu^*_h\mu^*_0)(m^*_2\rho(1-\pi)+m^*_3(1-(1-\pi)\rho))}$$

(5.9)

and the linearized Jacobian matrix evaluated at $E_0$ and $\beta_h^*$ denoted and given by

$$J(E_0, \beta_h^*) = \begin{pmatrix}
-\mu_1 & 0 & 0 & -m^*_1 & 0 & 0 & 0 & 0 & 0 & m^*_2 \\
0 & -m_2 & 0 & 0 & 0 & 0 & 0 & 0 & m^*_3 & 0 \\
0 & 0 & -m_3 & 0 & 0 & 0 & 0 & 0 & 0 & m^*_4 \\
0 & a & \sigma & -m_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma_5 & \gamma_6 & -m_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & m_7 & m_8 & -m_7 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

(5.10)
Where, 

\[ c_0 = 1, \]
\[ c_1 = m_1 + m_2 + m_5 \]
\[ c_2 = m_2m_5 + m_1(m_2 + m_5) - \frac{\phi^2 \omega \mu \nu}{\alpha \mu \nu} \]
\[ c_3 = m_1m_2m_5(1 - R_0^2) \]  

(5.12)

If we also substitute 1(one) for \( R_0 \) in to equation (5.11), then it will have a simple zero eigenvalue and the other eigenvalues have negative real parts. Therefore; the disease-free equilibrium point \( E_0 \) is a non-hyperbolic.

To compute the coefficients equation (5.11) and equation (5.12), we determine the right and left eigenvectors corresponding to the zero eigenvalue. Thus, the components of the right eigenvectors denoted by \( w_i \) for \( i = 1, ... , 7 \) are given by

\[
\begin{align*}
-\mu_h w_1 + \epsilon w_4 + \theta w_5 - m_1^2 w_7 &= 0 \\
-m_1 w_2 + m_2^2 w_7 &= 0 \\
-m_2 w_3 + m_3^2 w_7 &= 0 \\
\alpha w_2 + \alpha w_3 - m_3 w_4 &= 0 \\
\gamma w_2 + \gamma w_3 - m_4 w_5 &= 0 \\
-m_6 (w_2 + w_3) - \mu v_6 &= 0 \\
m_7 (w_2 + w_3) - m_5 w_7 &= 0 \\
\end{align*}
\]

(5.13)

Where, \( m_1^2 = \phi \omega \beta_h \), \( m_2^2 = \phi \omega \beta_h (1 - \pi) \), \( m_3^2 = \phi \omega \beta_h (1 - (1 - \pi) \rho) \), \( m_4^2 = m_7^2 = \frac{\phi \omega \mu \nu}{\alpha \mu \nu} \)

\[
\begin{align*}
w_3 &= \frac{\phi \omega \beta_h (\rho(m_2^2 (1 - \pi) \rho) - m_1 m_2 m_4)}{\mu_1 m_2 m_4} w_7, \quad w_2 = \frac{m_2^2}{m_1} w_7, \quad w_3 = \frac{m_3^2}{m_2} w_7, \quad w_4 = \frac{(m_1 m_2 m_3^2 + m_2 m_3 m_4^2)}{m_1 m_2 m_3} w_7, \\
w_5 &= \frac{\phi \omega \beta_h (\rho m_2^2 (1 - \pi) \rho) + \rho (1 - (1 - \pi) \rho))}{m_1 m_2 m_4} w_7, \quad w_6 = -\frac{m_6}{m_6} w_7, \quad w_7 = w_7 > 0 \\
\end{align*}
\]

(5.14)

And the components of the left eigenvectors denoted by \( v_i \), for \( i = 1, ... , 7 \) are given by

\[
\begin{align*}
-\mu v_1 &= 0 \\
-m_1 v_2 + \alpha v_4 + \gamma v_5 - m_6 (v_6 - v_7) &= 0 \\
-m_2 v_3 + \sigma v_4 + \gamma v_5 - m_5 (v_6 - v_7) &= 0 \\
\epsilon v_1 - m_3 v_4 &= 0 \\
\theta v_1 - m_4 v_5 &= 0 \\
-\mu v_6 &= 0 \\
-m_1^2 v_1 + m_2^2 v_2 + m_3^2 v_3 - m_5 v_7 &= 0 \\
v_1 = v_4 = v_5 = v_6 = 0, \quad v_7 = v_7 > 0, \quad v_2 = \frac{m_6}{m_1} v_7, \quad v_3 = \frac{m_6}{m_2} v_7 \\
\end{align*}
\]

(5.15)

(5.16)

Let we make the following change of state variables \( S_h = x_1, l_h = x_2, l_{hp} = x_3, T_h = x_4, R_h = x_5, S_p = x_6, l_p = x_7 \) and using the vector notation \( x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T \). The system (2.1) can then be written in the form \( \frac{dx}{dt} = F(x) \) where, \( F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \) as shown below
After substituting equation (5.14), equation (5.16) and equation (5.18) respectively into equation (5.20), then the simplified values of the coefficient $a$ in terms of $\omega$ and $\alpha$ is given by

$$a = \frac{v_2 w_2 \frac{\partial^2 f_2}{\partial \theta H}(x_0, 0) + v_3 w_2 \frac{\partial^2 f_2}{\partial \theta H}(x_0, 0) + v_4 w_2 \frac{\partial^2 f_2}{\partial \theta H}(x_0, 0) + w_s \frac{\partial^2 f_2}{\partial \theta H}(x_0, 0)}{\Lambda_H(\mu_0 + \delta + \gamma)} \nu_2 w_2^2 B_0$$

Where,

$$B_0 = \frac{\Lambda_H(m_2 \rho(1-\alpha)+m_1(1-(1-\alpha)\rho))}{\Lambda_H \mu^3} + \frac{\phi_0 \rho \mu_0 \Lambda_H \rho(1-(1-\alpha)\rho)}{\Lambda_H \mu^3} + \frac{\phi_0 \rho \mu_0 \Lambda_H \rho(1-(1-\alpha)\rho)m_2^2}{\Lambda_H \mu_0 m_2 m_4} + \frac{\phi_0 \rho \mu_0 \Lambda_H \rho(1-(1-\alpha)\rho)m_1 m_2 \rho(1-\alpha)}{\Lambda_H \mu_0 m_2 m_4}$$

After substituting equation (5.14), equation (5.16) and equation (5.19) respectively into equation (5.21), then the simplified values of the coefficient $b$ in terms of $w_7$ and $w_8$ is given by

$$b = \frac{v_2 w_2 \frac{\partial^2 f_2}{\partial \theta H}(x_0, 0) + v_3 w_2 \frac{\partial^2 f_2}{\partial \theta H}(x_0, 0)}{\Lambda_H(\mu_0 + \delta + \gamma)}$$
Clearly, the coefficient $b$ is positive since all the parameters are non-negative. Thus, the local dynamics of the system (1) around $E_0$, for $\beta_h = \beta_h^* \text{ is depends on the sign of the coefficient } a$. Similar to theorem [24] we also established the following theorem.

**Theorem 5** The system (1) will undergo backward bifurcation at $R_0 = 1$ if the coefficient $a$ is positive ($B_0 > 0$) or ($R_c < R_0$) otherwise it will exhibit a forward bifurcation if $a$ is negative ($B_0 < 0$).

### 5.2 Local stability of endemic equilibrium

**Theorem 3:** The endemic equilibrium point ($E^*$) of the system (2.1) is locally asymptotically stable if $R_0 > 1$.

Proof: The Jacobian matrix evaluated as

$$J(E^*) = \begin{pmatrix}
-P & 0 & 0 & \varepsilon & \theta & 0 & -Q \\
R & -m_1 & 0 & 0 & 0 & S \\
T & 0 & -m_2 & 0 & 0 & Z \\
0 & \alpha & \sigma & -m_3 & 0 & 0 \\
0 & \gamma_s & \gamma_r & 0 & \gamma_s & 0 \\
0 & -X & -X & 0 & 0 & -Y \\
0 & X & X & 0 & 0 & -m_5
\end{pmatrix} (5.22)$$

$$P = \omega\beta_h \frac{i_h}{N_h} + \mu_h, \quad R = \rho(1 - \pi)\omega\beta_h \frac{i_h}{N_h}, \quad T = (1 - (1 - \pi)\rho)\omega\beta_h \frac{i_h}{N_h}, \quad Q = \omega\beta_h \frac{i_h}{N_h}, \quad S = \rho(1 - \pi)\omega\beta_h \frac{i_h}{N_h}, \quad Z = (1 - (1 - \pi)\rho)\omega\beta_h \frac{i_h}{N_h}, \quad X = \omega\beta_h \frac{i_h}{N_h}(1 - \frac{\gamma_s + \gamma_r}{N_h})$$

$$Y = \omega\beta_h \frac{i_h}{N_h} + \mu_v$$

The eigenvalues of the $J(E^*)$ are given by:

Clearly $\lambda_1 = -\left(\omega\beta_h \frac{i_h}{N_h} + \mu_v\right) < 0$ and its associate characteristics equation is

$$\lambda^8 + e_1\lambda^7 + e_2\lambda^6 + e_3\lambda^5 + e_4\lambda^4 + e_5\lambda^3 + e_6\lambda^2 + e_7\lambda + e_8 = 0 \quad (5.23)$$

Where,

$$e_1 = P + m_2 + m_3 + m_4 + m_5$$
$$e_2 = m_2m_4 + (m_2 + m_3)(P - m_1 + m_2 + m_5) + (P + m_1)(m_1 + m_5)(m_3 + m_4) + XU$$
$$e_3 = m_2m_4 + (m_2 + m_3)(m_3m_4 + (P + m_1)(m_1 + m_5) + Pm_1 + Pm_3(m_3 + m_4)$$
$$+ (m_2m_5 - XZ)(P + m_1 + m_3 + m_4)$$
$$e_4 = m_2m_4(P + m_1 + m_2 + m_5 - XU) + (m_1 + m_3)(Pm_1 + (m_2 + m_3)Pm_1 + m_2m_5 - XZ) +$$
$$X(m_5 + XZ) + \theta_R R(1 + m_1) + \theta_R T(1 + m_1) + (m_1 + m_3)\theta(\gamma_s T + \gamma_r T) + \theta s(\sigma + m_1 + m_3) + Q(\gamma_r T - m_1 + \gamma_r R)m_2$$
$$e_5 = (m_2 + m_3)Pm_2m_4 + m_2m_4(m_2m_5 - XZ) + (m_1 + m_3)Pm_1(m_2m_5 - XZ) + eT(\sigma + m_1 + m_3)$$
$$+ eX(\gamma_r T + \gamma_r R) + \theta (m_1 + m_3)\gamma_r T - m_1 - \gamma_r R)m_2 + Q(\gamma_r T - m_1 - \gamma_r R)m_2 + \theta X(\gamma_r T + \gamma_r R)m_2$$
$$e_6 = Pm_2m_3m_5 + e_X(\gamma_r T - m_1 - \gamma_r R)m_2$$
$$e_7 = Pm_2m_3m_5 + e_X(\gamma_r T - m_1 - \gamma_r R)m_2$$
$$+ (m_1 + m_3)\theta(\gamma_r T - m_1 - \gamma_r R)m_2 + Q(\gamma_r T - m_1 - \gamma_r R)m_2$$
$$+ X(\gamma_r T + \gamma_r R)m_2$$

The Routh-Hurwitz criteria for Polynomial equation (5.23) will give six negative eigenvalues if the conditions given below are satisfied: $e_i > 0, \ for \ i = 1, 2, 3, . . . . 6.$ The relevant Routh Hurwitz criteria in [25] could be used to show that the model system (2.1) is stable locally asymptotically when $R_0 > 1$. 

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6 Global Stability

6.1 Global stability of disease-free equilibrium point

To investigate the global stability of the disease-free equilibrium point \( E_0 \), we consider the Lyapunov function [26]. So that

\[
V = m_5(m_2 l_{hs} + m_1 l_{hr}) l_h + \phi \omega \beta_h (\rho (1 - \pi) m_2 + (1 - (1 - \pi) \rho) m_3) l_p 
\]

After substituting \( \frac{dl_{hs}}{dt} = m_5 \left( m_2 \frac{dl_{hs}}{dt} + m_1 \frac{dl_{hr}}{dt} \right) \) and \( \frac{dl_{hr}}{dt} \) from (1) to (35) and simplifying it, then we get

\[
\frac{dV}{dt} = \frac{\phi^2 \omega^2 \beta_h \beta_p \phi (m_2 \rho (1 - \pi) + m_1 (1 - (1 - \pi) \rho)) (l_{hs} + l_{hr})}{l_h} - m_5 m_2 m_5
\]

After substituting \( \frac{dl_{hs}}{dt} = m_5 \frac{dl_{hr}}{dt} \) and \( \frac{dl_{hr}}{dt} \) from (1) to (35) and simplifying it, then we get

\[
\frac{dV}{dt} = \frac{\phi^2 \omega^2 \beta_h \beta_p \phi (m_2 \rho (1 - \pi) + m_1 (1 - (1 - \pi) \rho)) (l_{hs} + l_{hr})}{l_h} - m_5 m_2 m_5
\]

Since \( \frac{dS_h}{dt} \leq \frac{(1 - \pi) \lambda_h}{\mu_h} \) and \( \frac{dS_h}{dt} \leq \frac{\lambda_h \mu_h}{\mu_h} = \frac{\lambda_h \mu_h}{\mu_h} \) (36) is equivalent to

\[
\frac{dV}{dt} = \frac{\phi^2 \omega^2 \beta_h \beta_p \phi (m_2 \rho (1 - \pi) + m_1 (1 - (1 - \pi) \rho)) (l_{hs} + l_{hr})}{l_h} - m_5 m_2 m_5
\]

Since \( R_0 ^2 = \frac{\phi^2 \omega^2 \beta_h \beta_p \phi (m_2 \rho (1 - \pi) + m_1 (1 - (1 - \pi) \rho)) (l_{hs} + l_{hr})}{l_h} - m_5 m_2 m_5 \) (37) is also equivalent to

\[
\frac{dV}{dt} = m_5 m_2 m_5 (R_0 ^2 - 1)
\]

Therefore, \( \frac{dV}{dt} \leq 0 \) provided \( (R_0 ^2 - 1) \leq 0 \) which leads to \( R_0 \leq 1 \). \( \frac{dV}{dt} = 0 \) if and only if \( l_{hs} = l_{hr} = 0 \) or \( R_0 = 1 \).

Therefore, by Lasalle’s invariant principle [27], every solution to equations of the model system (2.1) with initial conditions in \( \Omega \) approaches to the disease-free equilibrium point \( E_0 \) at time \( t \) leads infinity whenever, \( R_0 \leq 1 \). Hence, the disease-free equilibrium \( E_0 \) is globally asymptotically stable if \( R_0 \leq 1 \).

**Theorem 6** The disease-free equilibrium point \( E_0 \) of system (2.1) is globally asymptotically stable if \( R_0 \leq 1 \) and unstable if \( R_0 > 1 \).

The epidemiological implication of theorem 6 is that the elimination of the malaria disease is possible regardless of initial condition system (2.2) of the sub-population of the model system (2.1) whenever \( R_0 \leq 1 \).

7 Analysis of the Model with Optimal Control

In this section, we consider model system (2.1) and incorporate optimal combinations of time dependent control measures namely, (i) prevention measure for drug resistance \( u_1(t) = u_1 \) to minimize the proportion of the emergence of drug resistant of malaria strains as well as spread of the disease dynamics. This includes improving the way drugs used through improving prescribing, follow up practices and patient compliance, (ii) the use of insecticide treated bed net (ITN) \( u_2(t) = u_2 \) as preventive measure i.e., to reduce the number of bites from mosquitoes as they physically provide a barrier between the infectious mosquitoes and the susceptible humans, and also to reduce the population of the mosquitoes by killing them after they land on the treated net. (iii) treatment with drugs \( u_3(t) = u_3 \), treating individuals who developed symptoms of the disease, and (iv) the use of indoor residual spray (IRS), \( u_4(t) = u_4 \) as preventive measure i.e., insecticide spray on the breeding site of mosquitoes reduces the number of mosquito populations by killing these rest indoors after feeding. The controls are practiced on time interval \([t_0, t_f]\), where \( t_0 \) and \( t_f \) are initial and final time respectively. After incorporating the above controls in to the basic model (2.1) we get the following modified state equations:
Here the following objective function $J$ is used to minimize the number of infected human with drug sensitive and drug resistance malaria parasite strains, infective in treatment human populations and total mosquito populations while keeping the costs of applying the controls $u_1, u_2, u_3$ and $u_4$ as low as possible.

$$J = \min_{U} \int_{0}^{T} \left( A_1 I_{hs} + A_2 I_{hr} + A_3 T_h + A_4 I_v + \frac{1}{2} \sum_{i=1}^{4} d_i u_i^2 \right) dt$$

(7.2)

Where, $i = 1,2,3,4$, $A_1, A_2, A_3$ and $A_4$ and $d_1, d_2, d_3$, and, $d_4$ are coefficients associated to the state variable and controls respectively. Following the approach [28,29], the cost of the controls have been chosen quadratic.

Thus, the goal is to find, an optimal control quadruple, $u_1^*, u_2^*, u_3^*$ and $u_4^*$ such that

$$J(u_1^*, u_2^*, u_3^*, u_4^*) = \min \{ J(u_1, u_2, u_3, u_4): u_1, u_2, u_3, u_4 \in U \}$$

(7.3)

Where, $U = \{ u_1(t) \ u_2(t) \ u_3(t) \ u_4(t) \ : 0 \leq u_i < 1, i = 1,2,...,4, 0 \leq t \leq t_f \}$ is the control set.

The Pontryagins’s Maximum Principle [29] converts the system (7.1) with equation (7.2) and equation (7.3) into a problem of minimizing pointwise the Hamiltonian $H$ with respect to $u_1, u_2, u_3$ and $u_4$

$$H = (S_h, I_{hs}, I_{hr}, T_h, R_h, S_v, I_v, t) = L(I_{hs}, I_{hr}, T_h, I_v, u_1, u_2, u_3, u_4, t) + \lambda_1 \frac{ds_h}{dt} + \lambda_2 \frac{dI_{hs}}{dt} + \lambda_3 \frac{dI_{hr}}{dt} + \lambda_4 \frac{dI_v}{dt} + \lambda_5 \frac{dT_h}{dt} + \lambda_6 \frac{dS_v}{dt} + \lambda_7 \frac{dI_v}{dt}$$

(7.4)

Where, $L(I_{hs}, I_{hr}, T_h, I_v, u_1, u_2, u_3, u_4, t) = A_1 I_{hs} + A_2 I_{hr} + A_3 T_h + A_4 I_v + \frac{1}{2} \sum_{i=1}^{4} d_i u_i^2$ for $i = 1,2,3,4$

and $\lambda_i$, for $i = 1,2,3,4,5,6,7$ are adjoint variable. Using the existence result for the optimal control [29], we established the following theorem as

**Theorem** There exists a set of an optimal control $u_i^* = (u_1^*, u_2^*, u_3^*, u_4^*)$ and corresponding state solution, $S_h^*, I_{hs}^*, I_{hr}^*, T_h^*, R_h^*, S_v^*$ and $I_v^*$ that minimizes $J(u_1, u_2, u_3, u_4)$ over $U$ subject to system (7.1). Further, there exists adjoin functions $\lambda_1(t), ... , \lambda_7(t)$, and $u_1(t), ... , u_4(t)$ satisfying

\[
\begin{align*}
\frac{d\lambda_1}{dt} &= \mu_h \lambda_1 + (1 - u_2) \lambda_6 (\lambda_4 - \rho (1 - (1 - u_3) \pi) \lambda_2 - (1 - (1 - (1 - u_1) \pi) \rho) \lambda_3) \\
\frac{d\lambda_2}{dt} &= -A_1 + \alpha (\lambda_2 - \lambda_4) - (\gamma_s + \tau_u) (\lambda_2 - \lambda_3) + (\mu_h + \delta_h) \lambda_2 + \frac{(1 - u_2) S_v}{n_h} \left( 1 - \frac{1}{n_h} \right) (\lambda_6 - \lambda_7) \\
\frac{d\lambda_3}{dt} &= -A_2 + \sigma (\lambda_3 - \lambda_4) - (\gamma_r + \tau_u) (\lambda_3 - \lambda_5) + (\mu_h + \delta_h) \lambda_3 + \frac{(1 - u_2) S_v}{n_h} \left( 1 - \frac{1}{n_h} \right) (\lambda_6 - \lambda_7) \\
\frac{d\lambda_4}{dt} &= -A_3 + (\epsilon + u_1 + (1 - \tau_u) \pi) (\lambda_4 - \lambda_1) + (\mu_h + \delta_h) \lambda_4 \\
\frac{d\lambda_5}{dt} &= \mu_h \lambda_5 + \theta (\lambda_5 - \lambda_1) \\
\frac{d\lambda_6}{dt} &= (1 - u_2) \lambda_6 (\lambda_6 - \lambda_7) + (\mu_v + \delta u_2 + \beta u_4) \lambda_6 \\
\frac{d\lambda_7}{dt} &= (1 - u_2) \frac{\phi \mu_h S_v}{n_h} (\lambda_4 - \rho (1 - (1 - u_1) \pi) \lambda_2 - (1 - (1 - (1 - u_1) \pi) \rho) \lambda_3) + (\mu_v + \delta_v + \delta u_2 + \beta u_4) \lambda_7 - A_4
\end{align*}
\]
with transversality conditions

$$\lambda_i(t_f) = 0 \text{ for } i = 1, 2, 3, 4, 5, 6, 7$$  \hspace{1cm} (7.6)$$

Further, the optimal controls $u_1^*, u_2^*, u_3^*$ and $u_4^*$ are given by

$$u_1^* = \min \left\{ \max \left( 0, \frac{(1-u_2)\rho \lambda_3 (\lambda_3-\lambda_2 + (\lambda_4-\lambda_1) \gamma_h)}{d_1}, 1 \right) \right\}$$

$$u_2^* = \min \left\{ \max \left( 0, \frac{\lambda_3 (1-u_1) \rho \lambda_3 (\lambda_3-\lambda_2 + (\lambda_4-\lambda_1) \gamma_h)}{d_2}, 1 \right) \right\}$$

$$u_3^* = \min \left\{ \max \left( 0, \frac{\tau (\lambda_3-\lambda_4) \gamma_h + \tau (\lambda_2-\lambda_4) \delta_h}{d_3}, 1 \right) \right\}$$

$$u_4^* = \min \left\{ \max \left( 0, \frac{\beta(S_u \lambda_k + l u \lambda_2)}{d_4}, 1 \right) \right\}$$  \hspace{1cm} (7.7)$$

Proof:

The existence of the optimal control follows from Fleming and Rischel [30] due to convexity of the integrand objective functional $J$ in (7.2) with respect to $u_i$, $i = 1, 2, 3, 4$ over the convex and closed control set $U$ and the system (7.1) satisfies the and Lipchitz property with respect to state variables since the state solutions are bounded. The differential equation (7.5) governing the adjoint variables $\lambda_1, \lambda_2, \ldots, \lambda_7$ are obtained by partial differentiation of the Hamiltonian $H$ equation (7.4) with respect to the corresponding state variables that is,

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial \delta_h}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial \delta_r}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial \gamma_h}, \quad \frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial \gamma_r}, \quad \frac{d\lambda_5}{dt} = -\frac{\partial H}{\partial \delta_r}, \quad \frac{d\lambda_6}{dt} = -\frac{\partial H}{\partial \delta_h}, \quad \frac{d\lambda_7}{dt} = -\frac{\partial H}{\partial \delta_h} \text{ with terminal conditions equation (7.6).}$$

The characterization of optimal control given by system (7.7) is obtained by partial derivative of the Hamiltonian $H$ equation (7.4) with respect to each control $u_i$ and solving $\frac{\partial H}{\partial \delta_i} = 0$, for $i = 1, 2, 3, 4$.

8 Numerical Simulation

In this section, numerical simulations are performed to confirm with our analytical results stated in the optimality system which is charachterized by the state system (7.1) and the adjoint system (7.5) was solved numerically by applying Runge Kutta fourth order schemes of the approach [31]. The implimentation of the scheme was done using MATLAB package.

The parameters values provided in Table 1 are used so that $R_0 = 1.1937255489 > 1$. The simulations of the model are done by using the initial conditions given by $S_1(0) = 800$, $I_{hr}(0) = 30$, $I_{hr}(0) = 30$, $T_1(0) = 30$, $R_1(0) = 10$, $S_p(0) = 5000$, $I_p(0) = 100$. To minimize malaria infectious humans and the total mosquito populations as well as minimizing the associated costs of controls, the weights constant values in the objective function (38) are chosen so that $A_1 = A_2 = A_3 = A_4 = d_1 = d_2 = d_3 = d_4 = d_2$.

In order to analyze the numerical results, we proposed optimal combinations of the aforementioned control strategies as alternative choose to minimize the spread of malaria disease dynamics. So as to do this, we introduced different optimal combination strategies in our model and numerically compare their effects on malaria infected populatons.Thus, the proposed optimal combinations and numerical result analysis are as follows

- **Strategy a**: Combination of use of preventive control of drug resistance, insecticide treated net ITN and treatment of infective individuals
- **Strategy b**: Combination of use of preventive control of drug resistance, indoor residual spray IRS for vector control and treatment of infective individuals
- **Strategy c**: Combination of use of insecticide treated nets ITN, indoor residual spray IRS for vector control and treatment of infective individuals
- **Strategy d**: Combination of use of preventive control of drug resistance, insecticide treated nets ITN and indoor residual spray IRS and treatment of infective individuals
Table 1. Lists of parameters of the model system (2.1)

<table>
<thead>
<tr>
<th>Parameter symbol</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_h$</td>
<td>0.8333</td>
<td>[32]</td>
</tr>
<tr>
<td>$\mu_h$</td>
<td>0.00005447</td>
<td>[33]</td>
</tr>
<tr>
<td>$\delta_h$</td>
<td>0.0680</td>
<td>[34]</td>
</tr>
<tr>
<td>$\gamma_s$</td>
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<td>[35]</td>
</tr>
<tr>
<td>$\gamma_r$</td>
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<td>[36]</td>
</tr>
<tr>
<td>$\tau$</td>
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<td>Assumed</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.2000</td>
<td>[37]</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.5020</td>
<td>[37]</td>
</tr>
<tr>
<td>$\Lambda_r$</td>
<td>0.0710</td>
<td>[38]</td>
</tr>
<tr>
<td>$\delta_r$</td>
<td>0.0100</td>
<td>[39]</td>
</tr>
<tr>
<td>$\mu_r$</td>
<td>0.0500</td>
<td>[40]</td>
</tr>
<tr>
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<td>$\theta$</td>
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<td>[41]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.7000</td>
<td>[42]</td>
</tr>
<tr>
<td>$\beta_r$</td>
<td>0.48</td>
<td>[41]</td>
</tr>
<tr>
<td>$\beta$</td>
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<tr>
<td>$\delta$</td>
<td>0.2500</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.0500</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>0.0500</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\pi$</td>
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<td>Assumed</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.0500</td>
<td>Assumed</td>
</tr>
</tbody>
</table>

8.1 Strategy a

Control with the preventive of drug resistance, insecticide treated net ITN, and treatment ($u_1 \neq 0$, $u_2 \neq 0$, $u_3 \neq 0$, $u_4 = 0$). In this strategy, we compare the strategy a situation where no control ($u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$) was used with the application of strategy a. It can be seen from the Figs. 3a, 3b, 3c, 3d, 3e and 3f that there is a significant increase in the number of susceptible and recovered human populations and a significant decrease in the number of infected with drug sensitive strains, infected with drug resistant strains, infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. From this, one can observe that strict application of strategy a for a period between 10 and 30 days is sufficient to reduce the number of individuals with malaria symptoms and malaria infected vectors to zero. It can be noted that, a combination of preventive of drug resistance, insecticide treated nets ITN, and treatment can play an important role in minimizing malaria infectious. The control profile shown in Fig. 3g shows that, controls $u_1$, $u_2$ and $u_3$ decreases from the maximum of 100% to the lower bound. This suggest that, a high effort is required for preventive control of drug resistance $u_1$, insecticide treated net ITN $u_2$ and medical treatment $u_3$ of individuals under this strategy.

8.2 Strategy b

Control with the preventive of drug resistance, indoor residual spray IRS, and treatment ($u_1 \neq 0$, $u_2 = 0$, $u_3 \neq 0$, $u_4 = 0$). In this strategy, we compare the strategy a situation where no control ($u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 0$) was used with the application of strategy b. It can be seen from the Figs. 4a,4b,4c,4d,4e and 4f that there is a significant increase in the number of susceptible and recovered human populations and a significant decrease in the number of infected with drug sensitive strains, infected with drug resistant strains, infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. Even though this strategy minimizes the number of malaria infectious populations, however, it is not enough to eliminate the disease at a given time and hence there is a need for additional control effort to eliminate the disease out of the community. The control profile shown in Fig. 4g shows that, controls $u_1$, $u_3$ and $u_4$ decreases from the maximum of 100% to the lower bound. This suggest that, a high effort is required for preventive control of drug resistance $u_1$, indoor residual spray IRS $u_4$ and medical treatment $u_3$ of individuals under this strategy.
Simulations of the model Showing the effect of preventive control of drug resistance, insecticide treated net ITN and Treatment controls

8.3 Strategy c

Control with insecticide treated net ITN, indoor residual spray IRS, and treatment ($u_1 = 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$) In this strategy, we compare the strategy a situation where no control ($u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0$) was used with the application of strategy c. It can be seen from the Figs. 5a,5b,5c,5d,5e and 5f that there is a significant increase in the number of susceptible and recovered human populations and a dramatic decrease in the number of infected with drug sensitive strains, infected with drug resistant strains, infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. With the application of strategy c, $I_{hr}$, $T_h$ and $I_r$ within time $t = 10$ days. $I_{hr}$ within time $t = 30$ days will be eliminated from the system. This result is a bit more promising than strategy a and strategy b. The control profile shown in Fig. 5g shows that, control $u_3$ is at 50% initially and decreases from the maximum of 70% to the lower bound while controls $u_2$ and $u_4$ decreases from the maximum of 100% to the lower bound within 90 days. This suggests that, a high effort is required for the use of insecticide treated net $u_2$, and indoor residual spray IRS $u_4$ for vector control and there is a low effort for the use of medical treatment $u_3$ of individuals under this strategy.

8.4 Strategy d

Control with the preventive of drug resistance, insecticide treated net ITN, indoor residual spray IRS, and treatment ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0, u_4 \neq 0$). In this strategy, we compare the strategy a situation where no
control \( u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0 \) was used with the application of strategy \( d \). It can be seen from the Figs. 6a,6b,6c,6d,6e and 6f that there is a significant increase in the number of susceptible and recovered human populations and a significant decrease in the number of infected with drug sensitive strains, infected with drug resistant strains, infective in treatment human populations and infected mosquito populations compared to the strategy with no control at a given time respectively. With the application of strategy \( d \), \( I_p \), \( I_{RS} \), \( T_h \) and \( I_{HR} \) within time \( t = 8, 10, 11 \) and 30 days respectively will be eliminated from the system. This result is a bit more promising than when strategy \( a \) and strategy \( b \) except possibly strategy \( c \) which yield almost the same results. The control profile shown in Fig. 6g shows that, controls \( u_1 \), \( u_2 \) and \( u_4 \) decreases from the maximum of 100% to the lower bound. This suggest that, a high effort is required for preventive control of drug resistance \( u_1 \), insecticide treated net \( u_2 \), indoor residual spray IRS \( u_4 \) and medical treatment \( u_3 \) of individuals under this strategy.

Fig. 4. Simulations of the model Showing the effect of preventive control of drug resistance, indoor residual spray IRS and treatment controls
Fig. 5. Simulations of the model showing the effect of insecticide treated net, indoor residual spray IRS and treatment controls

Fig. 6. Simulations of the model Showing the effect of preventive control of drug resistance, insecticide treated net, indoor residual spray and treatment controls
9 Discussions and Conclusions

In this study, a non-linear system of ordinary differential equation model that describes the dynamics of malaria disease transmission is formulated and analyzed. Conditions are derived from the existence of disease-free and endemic equilibria. The basic reproduction number $R_0$ of the model is obtained, and we investigated that it is a threshold parameter between the extinction and persistence of the disease. If $R_0$ is less than unity, then the disease-free equilibrium point is both locally and globally asymptotically stable resulting in the disease removing out of the host populations. The disease can persist whenever $R_0$ is greater than unity. Furthermore, at $R_0$ is equal to unity, existence conditions are derived from the endemic equilibrium for both forward and backward bifurcations.

The numerical simulations of the optimality system which is characterized by the state system (7.1) and the adjoint system (7.5) was solved numerically by applying Runge Kutta fourth order schemes. The result of numerical simulations of these can be seen from the Fig. 6 that the combination of prevention of drug resistance, insecticide treated net ITN, indoor residual spray IRS and active treatment or strategy d performs the best to control the disease in given time period of intervention. Finally, we note that with the strict application of either one of the incorporated combinations of optimal control strategies, it is possible to reduce the number populations with malaria symptoms to zero in the given time and the spread of the disease. Furthermore, we note that, application of optimal control strategy is not only reduce the number populations with malaria symptoms but also it reduces the emergence of drug resistant malaria strains as well as the spread of the disease.

10 Recommendations

Here we recommend to malaria control policy makers, health care workers and any concerning body may use the incorporated strategy in this paper to dwindle the malaria disease burden on the community.

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Completing Interests

Authors have declared that no completing interests exist.

References


