Congruences of Partial Sum of Binomial Coefficients

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Let $p > 3$ be a prime, using the constant term method and the binomial coefficient expansion, we obtained the congruences of partial sum of binomial coefficients modulo $p^2$,

$$\sum_{i=1}^{\left\lfloor \frac{p+1}{3} \right\rfloor} \binom{p}{3i-2} \equiv \sum_{i=1}^{\left\lfloor \frac{p}{3} \right\rfloor} \binom{p}{3i-1} \equiv \sum_{i=1}^{\left\lfloor \frac{p}{3} \right\rfloor} \binom{p}{3i} \pmod{p^2}.$$

Keywords: Binomial coefficient; congruence; constant term method; power series expansion.

1 Introduction

The congruence property of binomial coefficient sum is an important research subject in mathematics. In 1878, Lucas [1] proved the congruence theorem of binomial coefficients. Let $n_0$, $k_0$ be the remainder of $n$, $k$ with respect to modulo $p$, then

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\[
\binom{n}{k} \equiv \left\lfloor \frac{n}{p} \right\rfloor \binom{n_0}{k_0} \pmod{p},
\]

where \(\lfloor x \rfloor\) is the greatest integer not greater than \(x\). More generally, for any positive integers \(n\) and \(k\), if

\[
n = n_0 + n_1 p + n_2 p^2 + \cdots + n_d p^d, k = k_0 + k_1 p + k_2 p^2 + \cdots + k_d p^d,
\]

are the base \(p\) expansions of \(n\) and \(k\) respectively, where \(0 \leq n_i, k_i \leq p - 1\), then

\[
\binom{n}{k} \equiv \binom{n_0}{k_0} \binom{n_1}{k_1} \cdots \binom{n_d}{k_d} \pmod{p}.
\]

In 2006, Sun Zhiwei and Pan Hao [2] proved that for any prime number \(p > 3\), then

\[
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left\lfloor \frac{p}{3} \right\rfloor \pmod{p},
\]

where \(\left\lfloor \cdot \right\rfloor\) is the Legendre Symbol. In 2011, Sun Zhiwei and Tauraso [3] extended and proved the congruence modulo \(p^2\),

\[
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left\lfloor \frac{p}{3} \right\rfloor \pmod{p^2},
\]

and

\[
\sum_{k=0}^{p-1} \frac{1}{k+1} \binom{2k}{k} \equiv \frac{3}{2} \left\lfloor \frac{p}{3} \right\rfloor - \frac{1}{2} \pmod{p^2},
\]

where \(p \geq 5\). Similar congruence on sums of combinatorial sequences have been studied in [4].

q-Analogues of congruences were also widely discussed by several authors (see, for instance, [5–9]).

Apagodu and Zeiberger [10] conjectured that for any prime \(p > 3\) and any positive integer \(r\),

\[
\sum_{k=0}^{p-1} \frac{2k}{k} \equiv \begin{cases} 
\alpha_r \pmod{p^2}, & \text{if } p \equiv 1 \pmod{3}, \\
-\alpha_r \pmod{p^2}, & \text{if } p \equiv 2 \pmod{3},
\end{cases}
\]

where

\[
\alpha_r = \sum_{k=0}^{r-1} \binom{2k}{k}.
\]

\[
\frac{1}{n} \left( \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k} - \frac{p}{3} \sum_{k=0}^{\frac{n-1}{2}} \binom{2k}{k} \right) \equiv 0 \pmod{p^2}.
\]

Based on the research of Sun Zhiwei and Tauraso [3], the following conclusions are obtained:

**Theorem 1**: Let \( p \) be an odd prime number, if \( p \equiv 1 \pmod{3} \), we have
\[
\sum_{i=1}^{\frac{p}{3}} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\frac{p}{3}} \binom{2p}{3i-1} \pmod{p^2}.
\]
If \( p \equiv -1 \pmod{3} \), we have
\[
\sum_{i=1}^{\frac{p+1}{3}} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\frac{p+1}{3}} \binom{2p}{3i-1} \pmod{p^2}.
\]

2 Preliminaries

**Definition 1** [13]: (Constant term method) Given Laurent polynomial \( P(x_1, x_2, \ldots, x_n) \), define

\[
CT \ P(x_1, x_2, \ldots, x_n)
\]

is the constant term of \( P(x_1, x_2, \ldots, x_n) \) and define

\[
COEEF \ \left[ x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \right] P(x_1, x_2, \ldots, x_n)
\]

is the coefficient of \( x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \).

**Lemma 1**[3]: Let \( p \) be an odd prime number, then
\[
\sum_{n=0}^{\frac{p-1}{2}} \binom{2n}{n} \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } p = 3, \\ 1 \pmod{p^2}, & \text{if } p \equiv 1 \pmod{3}, \\ -1 \pmod{p^2}, & \text{if } p \equiv -1 \pmod{3}. \end{cases}
\]

**Lemma 2**[14]: For any real number \( a \), we have
\[
\sum_{i=1}^{n} a q^{i-1} = \begin{cases} a(1 - q^n), & \text{if } q \neq 1, \\ 1 - q, & \text{if } q = 1, \end{cases}
\]

and
\[
\sum_{n=1}^{\infty} aq^{n-1} = \begin{cases} 
\infty, & \text{if } |q| \geq 1, \\
\frac{a}{1-q}, & \text{if } |q| < 1.
\end{cases}
\]

3 Proof of Theorem 1

By Definition 1, we have
\[
\sum_{n=0}^{p-1} \binom{2n}{n} \frac{CT}{x^{p-1}} \left[ \frac{1}{x} \right]^{p} = CT \sum_{n=0}^{\infty} \left( \frac{(1+x)^{2n}}{x} \right)^{n-1}.
\]
(1)

By Lemma 2, we obtain
\[
\sum_{n=0}^{\infty} \left[ \frac{(1+x)^{2n}}{x} \right]^{n} = \frac{1 - \left( \frac{(1+x)^{2}}{x} \right)^{p}}{1 - \frac{(1+x)^{2}}{x}}.
\]
(2)

By (1) and (2), we get
\[
\sum_{n=0}^{p-1} \binom{2n}{n} \frac{CT}{x^{p-1}} = \frac{1 - \left( \frac{(1+x)^{2}}{x} \right)^{p}}{1 - \frac{(1+x)^{2}}{x}}.
\]
(3)

Multiply \(x^p\) on the numerator and denominator of the right-hand side of equation (3), then
\[
\sum_{n=0}^{p-1} \binom{2n}{n} \frac{CT}{x^{p-1}} = \frac{x^p - (1+x)^{2p}}{x^p - (1+x)^{2p} x^{p-1}} \left[ x - (1+x)^2 \right] = CT \frac{(1+x)^{2p} - x^p}{(1+x+x^2)x^{p-1}}.
\]
(4)

Since
\[
1 + x + x^2 = \frac{1-x^3}{1-x}.
\]
(5)

Substitute (5) into (4) to get
\[
\sum_{n=0}^{p-1} \binom{2n}{n} \frac{CT}{x^{p-1}} = \frac{(1+x)^{2p} - x^p}{(1-x^3)x^{p-1}}.
\]
(6)

When \(|x| < 1\), we have
\[
\frac{1}{1-x^3} = \sum_{i=0}^{\infty} x^{3i}.
\]
(7)
By (6) and (7), we get

\[
\sum_{n=0}^{p-1} \binom{2n}{n} = CT \frac{1}{x^{p-1}} \left[ (1+x)^{2p} - x^{p} \right] \left[ (1-x) \sum_{i=0}^{\infty} x^{3i} \right] \\
= CT \frac{1}{x^{p-1}} \left[ (1+x)^{2p} - x^{p} \right] \left( \sum_{i=0}^{\infty} x^{3i} - \sum_{i=0}^{\infty} x^{3i+1} \right) \\
= COEEF_{\left[ 1 \right]^{p-1}} \left[ (1+x)^{2p} - x^{p} \right] \left( \sum_{i=0}^{\infty} x^{3i} - \sum_{i=0}^{\infty} x^{3i+1} \right).
\]

(8)

When \( p \equiv 1 \pmod{3} \), we have

\[
COEEF_{\left[ 1 \right]^{p-1}} \left[ (1+x)^{2p} - x^{p} \right] \left( \sum_{i=0}^{\infty} x^{3i} - \sum_{i=0}^{\infty} x^{3i+1} \right) \\
= 1 - \binom{2p}{2} + \binom{2p}{3} - \binom{2p}{5} + \binom{2p}{6} - \cdots - \binom{2p}{p-2} + \binom{2p}{p-1}. 
\]

(9)

By Lemma 1, when \( p \equiv 1 \pmod{3} \), we have

\[
\sum_{n=0}^{p-1} \binom{2n}{n} \equiv 1 \pmod{p^2}. 
\]

(10)

Combined (8), (9) and (10)

\[
1 - \binom{2p}{2} + \binom{2p}{3} - \binom{2p}{5} + \binom{2p}{6} - \cdots - \binom{2p}{p-2} + \binom{2p}{p-1} \equiv 1 \pmod{p^2}. 
\]

Then

\[
\sum_{i=1}^{\left[ \frac{p}{3} \right]} \binom{2p}{3i} \equiv \sum_{i=1}^{\left[ \frac{p}{3} \right]} \binom{2p}{3i-1} \pmod{p^2}. 
\]

(11)

When \( p \equiv -1 \pmod{3} \), we have

\[
COEEF_{\left[ 1 \right]^{p-1}} \left[ (1+x)^{2p} - x^{p} \right] \left( \sum_{i=0}^{\infty} x^{3i} - \sum_{i=0}^{\infty} x^{3i+1} \right) \\
= -1 + \binom{2p}{1} - \binom{2p}{3} + \binom{2p}{4} - \binom{2p}{6} + \cdots - \binom{2p}{p-2} + \binom{2p}{p-1}. 
\]

(12)

By Lemma 1, when \( p \equiv -1 \pmod{3} \), we have

\[
\sum_{n=0}^{p-1} \binom{2n}{n} \equiv -1 \pmod{p^2}. 
\]

(13)
Combined (11)-(13), we have

\[-1 + \binom{2p}{1} - \binom{2p}{3} + \binom{2p}{4} - \binom{2p}{6} + \cdots - \binom{2p}{p-2} + \binom{2p}{p-1} \equiv -1 \pmod{p^2}.

Then

\[\sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lceil \frac{p}{3} \rceil} \binom{2p}{3i} \pmod{p^2}. \tag{14}\]

4 Conclusion

By proof of Theorem 1, if prime \( p > 3 \), we have the congruence

\[\sum_{i=1}^{\lceil \frac{p}{3} \rceil} \binom{2p}{3i-1} \equiv \sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i-2} \pmod{p^2}, \text{ if } p \equiv 1 \pmod{3},
\]

\[\sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lceil \frac{p}{3} \rceil} \binom{2p}{3i} \pmod{p^2}, \text{ if } p \equiv -1 \pmod{3}.
\]

We wonder whether the congruence

\[\sum_{i=1}^{\lceil \frac{p}{3} \rceil} \binom{2p}{3i} \equiv \sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i-1} \pmod{p^2}, \text{ if } p \equiv 1 \pmod{3},
\]

\[\sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lceil \frac{p}{3} \rceil} \binom{2p}{3i} \pmod{p^2}, \text{ if } p \equiv -1 \pmod{3}.
\]

is established?

It's not hard to prove that it is also true. Since

\[\sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lceil \frac{p}{3} \rceil} \frac{2p(2p-1)(2p-2)\cdots(2p-3i+3)}{(3i-2)!} \equiv 2p \sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \frac{(-1)^{3i-3}}{3i-2} \pmod{p^2},
\]

\[\sum_{i=1}^{\lceil \frac{p}{3} \rceil} \binom{2p}{3i-1} \equiv \sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \frac{2p(2p-1)(2p-2)\cdots(2p-3i+1)}{(3i)!} = 2p \sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \frac{(-1)^{3i-1}}{3i} \pmod{p^2},
\]

\[\sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i} \equiv \sum_{i=1}^{\lceil \frac{p}{3} \rceil} \frac{2p(2p-1)(2p-2)\cdots(2p-3i+2)}{(3i-1)!} = 2p \sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \frac{(-1)^{3i-2}}{3i-1} \pmod{p^2}.
\]

When \( p \equiv 1 \pmod{3} \),

\[\sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lceil \frac{p}{3} \rceil} \binom{2p}{3i-1} \pmod{p^2}.
\]

\[\sum_{i=1}^{\lceil \frac{p}{3} \rceil} \binom{2p}{3i} \equiv \sum_{i=1}^{\lceil \frac{p+1}{3} \rceil} \binom{2p}{3i} \pmod{p^2}.
\]


\[ \sum_{i=1}^{\frac{p+1}{3}} (-1)^{3i-3} \frac{3i-2}{3i} \equiv \sum_{i=1}^{\frac{p}{3}} (-1)^{3i-1} \frac{3i}{3} \pmod{p}, \]

this is because the i-th item on the left is congruent to the reciprocal i-th item on the right with respect to the modulo \( p \), thus

\[ \sum_{i=1}^{\frac{p+1}{3}} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\frac{p}{3}} \binom{2p}{3i} \pmod{p^2}. \]

When \( p \equiv -1 \pmod{3} \),

\[ \sum_{i=1}^{\frac{p}{3}} (-1)^{3i-2} \frac{3i-1}{3i} \equiv \sum_{i=1}^{\frac{p}{3}} (-1)^{3i-1} \frac{3i}{3} \pmod{p}, \]

this is also because the i-th item on the left is congruent with the reciprocal i-th item on the right with respect to the modulo \( p \), so

\[ \sum_{i=1}^{\frac{p}{3}} \binom{2p}{3i} \equiv \sum_{i=1}^{\frac{p}{3}} \binom{2p}{3i-1} \pmod{p^2}. \]

In conclusion, if \( p > 3 \) is a prime, then we obtain the congruences of partial sum of binomial coefficients classified by 3 modulo \( p^2 \),

\[ \sum_{i=1}^{\frac{p+1}{3}} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\frac{p}{3}} \binom{2p}{3i-1} \equiv \sum_{i=1}^{\frac{p}{3}} \binom{2p}{3i} \pmod{p^2}. \]

In the future, we want to get more congruences of partial sum of binomial coefficients classified by \( k \) modulo \( p^\alpha(\alpha \geq 2) \).

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**Competing Interests**

Authors have declared that no competing interests exist.

**References**


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