Determinants and Inverses of Skew Symmetric Generalized Toeplitz Matrices

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Abstract

In this paper, explicit determinants and inverses of skew symmetric generalized Toeplitz matrices are given by constructing the special transformation matrices.

Keywords: generalized Toeplitz matrices; determinant; inverse; fibonacci number.

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1 Introduction

Toeplitz matrices have important applications in various disciplines including image processing, signal processing, and solving least squares problems [1]. It is an ideal research area and hot topic for the inverses of Toeplitz matrices and the special matrices with famous numbers. Due to the special structure, it is a major topic of research that the inversion of Toeplitz matrices can be reconstructed by use of a low number of its columns and the entries of the original Toeplitz matrix. The stability of the algorithms emerging from Toeplitz matrix inversion formulas was considered in [2].
In addition, some researchers showed the explicit determinants and inverses of the special matrices involving famous numbers. In [3], M. Akbulak and D. Bozkurt discussed originally Fibonacci and Lucas Toeplitz matrices with entries from Fibonacci and Lucas numbers, and they presented the upper and lower bounds for the spectral norms of the Fibonacci and Lucas Toeplitz matrix. The authors considered the determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [4]. In [5], circulant type matrices with the $k$-Fibonacci and $k$-Lucas numbers are presented and the explicit determinants and inverse matrices are presented by constructing the transformation matrices. Jiang et al. [6] gave the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and provided the determinants and the inverses of these matrices. And for the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Padovan, Perrin, Tribonacci and the generalized Lucas numbers by the inverse factorization of polynomial in [7].

It should be noted that Jiang and Zhou [8] obtained the explicit formula for spectral norm of an $r$-circulant matrix whose entries in the first row are alternately positive and negative, and the authors [9] investigated explicit formulas of spectral norms for $g$-circulant matrices with Fibonacci and Lucas numbers. The authors [10] proposed the invertibility criterium of the generalized Lucas skew circulant type matrices and provided their determinants and the inverse matrices. Furthermore, in [11] the determinants and inverses are discussed and evaluated for Tribonacci skew circulant type matrices.

In this paper, we will show the explicit determinants and inverses of the skew symmetric generalized Toeplitz matrices.

Here the Fibonacci sequence is defined by the following recurrence relation:

$$F_{n+1} = F_n + F_{n-1} \quad (n \geq 1) \quad \text{where} \quad F_0 = 0, \quad F_1 = 1.$$  

**Definition 1.1.** An $n \times n$ skew symmetric generalized Toeplitz matrix is meant a square matrix of the form

$$T_{F_k,n} = \begin{pmatrix} 
0 & F_k & F_{k+1} & \cdots & F_{k+n-3} & F_{k+n-2} \\
-F_k & 0 & F_k & \cdots & F_{k+n-4} & F_{k+n-3} \\
-F_{k+1} & -F_k & 0 & \cdots & F_{k+n-5} & F_{k+n-4} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-F_{k+n-3} & -F_{k+n-4} & -F_{k+n-5} & \cdots & 0 & F_k \\
-F_{k+n-2} & -F_{k+n-3} & -F_{k+n-4} & \cdots & -F_k & 0 
\end{pmatrix}_{n \times n},$$  

where $F_k, F_{k+1}, \cdots, F_{k+n-2}$ are the Fibonacci numbers, and $k \geq 2$.

Obviously, this matrix is completely determined by its first row, and $T_{F_k,n} = -T_{F_k,n}$.

Specially, in the case of $k = 2$, explicit determinants and inverses of Fibonacci skew symmetric Toeplitz matrices are given in [12].

## 2 Determinants and Inverses of the Skew Symmetric Generalized Toeplitz Matrices

In this section, we will give the determinant and the inverse of the matrix $T_{F_k,n}$.

Obviously, the determinant of an $n$-dimension skew symmetric matrix is zero, if $n$ is an odd number. So in this essay we always assume that $n$ is an even number.
Theorem 2.1. Let $T_{F_k,n}$ be a skew symmetric generalized Toeplitz matrix as the form of (1.1), we have

$$\det T_{F_k,n} = F_{k+n-2} \beta_1 \det G_{n-2}(\beta_i^t_{i=0} = F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) - F_{k-1} \beta_1 \det G_{n-2}(\beta_i^t_{i=0} = F_{k+n-3}, -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1})$$

where

$$G_{n-2}(\beta_i^t_{i=0} = F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1})$$

\begin{align*}
\begin{array}{cccccccc}
\beta_2 & \beta_3 & \beta_4 & \cdots & \cdots & \cdots & \beta_{n-3} & \beta_{n-2} \\
2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
F_{k+2} & 2F_k & F_{k-1} & & & & & \\
F_{k-1} & -2F_k & 0 & 2F_k - F_{k-1} & & & & \\
F_{k-1} & F_{k-1} - 2F_k & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \cdots & 0 & -F_{k-1} & F_{k-1} - 2F_k & 0 & 2F_k - F_{k-1} & F_{k-1} \\
\end{array}
\end{align*}

\[G_{n-2}(\beta_i^t_{i=0} = F_{k+n-3}, -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1})\]

\begin{align*}
\begin{array}{cccccccc}
F_{k+n-3} & F_{k+n-4} & F_{k+n-5} & \cdots & \cdots & \cdots & F_{k+2} & F_{k+1} \\
2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
F_{k+2} & 2F_k & F_{k-1} & & & & & \\
F_{k-1} & -2F_k & 0 & 2F_k - F_{k-1} & & & & \\
F_{k-1} & F_{k-1} - 2F_k & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \cdots & 0 & -F_{k-1} & F_{k-1} - 2F_k & 0 & 2F_k - F_{k-1} & F_{k-1} \\
\end{array}
\end{align*}

\[|\beta_i^t_{i=0} = \beta_2, \beta_3, \cdots, \beta_{n-1}; [F_i^k]_{i=k+n-3} = F_{k+n-3}, F_{k+n-4}, \cdots, F_k, \]

\[
\alpha_1 = \sum_{i=0}^{n-2} F_i x_{n-i-1}, \quad \alpha_2 = -\sum_{i=0}^{n-3} F_{i+1} x_{n-i-1}, \quad \beta_1 = \sum_{i=0}^{n-4} (-F_{k+n-4} + \frac{F_{k+n-3} F_{k+n-3}}{F_{k+n-2}}) x_{n-i-1} + \frac{F_k F_{k+n-3}}{F_{k+n-2}} x_2 + F_k x_1, \]

\[
\beta_2 = \frac{F_k F_{k+n-3}}{F_{k+n-2}}, \quad \beta_i = -F_{k+i-3} + \frac{F_k F_{k+n-3}}{F_{k+n-2}}, \quad (i = 3, 4, \cdots, n-1), \]

\[
x_1 = 1, \quad x_3 = \frac{x_2}{2}, \quad x_{k+1} x_3 + 2F_k x_2 + F_k + x_2 = 0, \quad (i = 4, 5, 6, \cdots, n-1),
\]

\[
x_i = -\frac{2F_k x_{i-1} + F_{k+2} x_{i-2} + \sum_{j=1}^{i-3} 2F_k x_{i-j-2}}{F_k}, \quad (i = 4, 5, 6, \cdots, n-1),
\]
\[ \det G_{n-2}(\beta_i^{n-1}, -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) = \]
\[ \begin{pmatrix} (-1)^{k-1}\beta_{k-1} \det L_{n-3}(-F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) + \\
F_{k-1} \det L_{n-3}(0, 2F_k - F_{k-1}, F_{k-1}) \end{pmatrix}, \]
\[ \det G_{n-2}(\beta_i^{n-2}, -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) = \]
\[ \begin{pmatrix} (-1)^{k-1}F_k \det L_{n-3}(-F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) + \\
F_{k-1} \det L_{n-3}(0, 2F_k - F_{k-1}, F_{k-1}) \end{pmatrix}. \]

Let \( L_i(-F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) = \)
\[ \begin{pmatrix} 2F_k & F_{k-1} & 0 & \cdots & \cdots & \cdots & 0 \\
F_{k+2} & 2F_k & F_{k-1} & \cdots & \cdots & \cdots \\
F_{k-1} - 2F_k & 0 & 2F_k - F_{k-1} & \cdots & \cdots & \cdots \\
-2F_k & F_{k-1} - 2F_k & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & F_{k-1} \end{pmatrix} \]
\[ \text{for } i = 1, 2, \ldots, n. \]

Let \( L_i(-F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) = \)
\[ \begin{pmatrix} F_k^2 & F_{k-1} & 0 & \cdots & \cdots & \cdots & 0 \\
F_{k+2}^2 & 2F_k & F_{k-1} & \cdots & \cdots & \cdots \\
F_{k-1} - 2F_k & 0 & 2F_k - F_{k-1} & \cdots & \cdots & \cdots \\
-2F_k & F_{k-1} - 2F_k & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & F_{k-1} \\
0 & \cdots & \cdots & \cdots & 0 & F_{k-1} - 2F_k & 0 & 2F_k - F_{k-1} \end{pmatrix} \]
\[ \text{for } i = 1, 2, \ldots, n. \]

Let \( L_1(-F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) = 2F_k, \)
\[ L_2(-F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) = 4F_k^2 - 2F_kF_{k-1} - F_{k-1}^2, \]
\[ L_3(-F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) = 8F_k^3 - 8F_k^2F_{k-1} - 2F_kF_{k-1}^2 + 2F_{k-1}^3, \]
\[ L_4(-F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) = 16F_k^4 - 24F_k^3F_{k-1} + 9F_kF_{k-1}^3 - 2F_{k-1}^4. \]

**Proof.** Let \( T_{F_k,n} \) be an \( n \times n \) skew symmetric generalized Toeplitz matrix. In the case \( n \geq 4 \), let

\[ \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots \\
1 & k & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix} \]

be two \( n \times n \) matrices, which are invertible. And \( x_i \) \((i = 1, 2, \ldots, n-1)\) are the same as (2.5)-(2.6).
Multiplying $T_{F_k,n}$ by $C_1$ from the left, and multiplying $D_1$ from the right, we obtain

$$C_1 T_{F_k,n} D_1 = \begin{pmatrix}
\alpha_1 \quad F_{k+n-3} & F_{k+n-4} & F_{k+n-5} & \cdots & F_{k+2} & F_k \\
\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 & \cdots & \beta_{n-3} \quad \beta_{n-2} \quad \beta_{n-1} \\
0 \quad 2F_k \quad F_{k-1} \quad 0 \quad \cdots \quad \cdots \quad 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 \quad 0 \quad 2F_{k+n-6} \quad 2F_{k+n-7} \quad 2F_{k+n-8} \quad \cdots \quad 2F_k \quad F_{k-1} \quad 0 \\
\end{pmatrix},$$

where $\alpha_1, \alpha_2, \beta_i \ (i = 1, 2, \ldots, n-1)$ are the same as (2.2)-(2.4), and from the last matrix we can easily get,

$$\det(C_1 T_{F_k,n} D_1) = F_{k+n-2} \det G_{n-1}(\alpha_1, [F_i]_{i=k+n-3}^k, [\beta_i]_{i=1}^{n-1}).$$

where

$$G_{n-1}(\alpha_1, [F_i]_{i=k+n-3}^k, [\beta_i]_{i=1}^{n-1}) = \begin{pmatrix}
\alpha_1 \quad F_{k+n-3} & F_{k+n-4} & F_{k+n-5} & \cdots & F_{k+2} & F_k \\
\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 & \cdots & \beta_{n-3} \quad \beta_{n-2} \quad \beta_{n-1} \\
0 \quad 2F_k \quad F_{k-1} \quad 0 \quad \cdots \quad \cdots \quad 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
0 \quad 0 \quad 2F_{k+n-6} \quad 2F_{k+n-7} \quad 2F_{k+n-8} \quad \cdots \quad 2F_k \quad F_{k-1} \quad 0 \\
\end{pmatrix}_{(n-1) \times (n-1)}.$$  

In order to simply compute the determinant of $G_{n-1}(\alpha_1, [F_i]_{i=k+n-3}^k, [\beta_i]_{i=1}^{n-1})$, we apply methods of elementary row transformation to this matrix, then we can obtain $G'_{n-1}(\alpha_1, [F_i]_{i=k+n-3}^k, [\beta_i]_{i=1}^{n-1})$. 
it is the form as,

\[ G_{n-1}(\alpha_1; [F_k]_{i=k+n-3}^{k}; [\beta_i]_{i=2}^{n-1}) \]

then we can obtain

\[
\begin{vmatrix}
\alpha_1 & r_{k+n-3} & r_{k+n-4} & r_{k+n-5} & \cdots & r_{k+2} & r_{k+1} & r_k \\
\beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_{n-3} & \beta_{n-2} & \beta_{n-1} \\
0 & 2r_k & r_{k-1} & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & r_{k+2} & 2r_k & r_{k-1} & \cdots & \vdots & \vdots & \vdots \\
\vdots & r_{k-1} - 2r_k & 0 & 2r_k - r_{k-1} & \cdots & \vdots & \vdots & \vdots \\
\vdots & 0 & r_{k-1} - 2r_k & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0 & -r_{k-1} & r_{k-1} - 2r_k \\
& 0 & \cdots & \cdots & \cdots & 0 & 2r_k - r_{k-1} & 0 \\
& & & & & & & \vdots & \vdots \\
& & & & & & & & 0 \\
\end{vmatrix}
\]

then we can obtain

\[
\det G_{n-1}(\alpha_1; [F_k]_{i=k+n-3}^{k}; [\beta_i]_{i=1}^{n-1}) = \\
\alpha_1 \det G_{n-2}(\beta_{i=2}^{n-1}; -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) - \\
\beta_1 \det G_{n-2}(F_{i=k+n-3}^{k}; -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}), \quad (2.14)
\]

where \( G_{n-2}(\beta_{i=2}^{n-1}; -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) \) and \( G_{n-2}(F_{i=k+n-3}^{k}; -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) \) are the forms as in the interpretation of Theorem 2.1.

To obtain the determinant of \( G_{n-2}(\beta_{i=2}^{n-1}; -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) \) and \( G_{n-2}(F_{i=k+n-3}^{k}; -F_{k-1}, F_{k-1} - 2F_k, 0, 2F_k - F_{k-1}, F_{k-1}) \), developed them in accordance with the last column, and in turn, we can get recursive formulas (2.7)-(2.13). And observe that

\[
\det \mathcal{C}_1 = \det \mathcal{D}_1 = (-1)^{(n-1)(n-2)}
\]

then we can obtain \( \det T_{F_k,n} \), which completes the proof.

\( \square \)

**Theorem 2.2.** Let \( T_{F_k,n} \) be a skew symmetric generalized Toeplitz matrix as the form of (1.1). If \( T_{F_k,n} \) is a nonsingular matrix, then

\[
T_{F_k,n}^{-1} = 
\begin{pmatrix}
0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_{n-1} & \gamma_n \\
-\gamma_1 & 0 & \gamma_2 & \gamma_3 & \gamma_4 & \cdots & \gamma_{n-1} & \gamma_n \\
-\gamma_2 & -\gamma_1 & 0 & \gamma_3 & \gamma_4 & \cdots & \gamma_{n-1} & \gamma_n \\
-\gamma_3 & -\gamma_2 & -\gamma_1 & 0 & \gamma_4 & \cdots & \gamma_{n-1} & \gamma_n \\
-\gamma_4 & -\gamma_3 & -\gamma_2 & -\gamma_1 & 0 & \cdots & \gamma_{n-1} & \gamma_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
-\gamma_{n-1} & -\gamma_{n-2} & -\gamma_{n-3} & -\gamma_{n-4} & -\gamma_{n-5} & \cdots & 0 & \gamma_1 \\
-\gamma_n & -\gamma_{n-1} & -\gamma_{n-2} & -\gamma_{n-3} & -\gamma_{n-4} & -\gamma_{n-5} & \cdots & \gamma_1 \\
\end{pmatrix}
\]

(2.15)
that is to say $T_{F_k,n}^{-1}$ is skew symmetric about diagonal and symmetric about secondary diagonal as well, where

$$
\begin{align*}
\gamma_{12} & = \epsilon_{1n}, \quad \gamma_{13} = \epsilon_{1n-1} + \epsilon_{1n}, \quad \gamma_{23} = \epsilon_{2n-3} + \epsilon_{2n}, \quad \gamma_{1n} = \epsilon_{1n} - \frac{F_{k+n-3}}{F_{k+n-2}} \epsilon_{13} - \epsilon_{14}, \\
\gamma_{ij} & = \epsilon_{i,n+2-j} + \epsilon_{i,n+3-j} - \epsilon_{i,n+j-3}, \ (i = 1, 2, \ldots, n; j = 4, 5, \ldots, n-1), \\
\epsilon_{11} & = - \frac{1}{F_{k+n-2}}, \quad \epsilon_{12} = \frac{g_1}{\alpha_1}, \quad \epsilon_{13} = \frac{g_2}{\zeta_1} + \sum_{i=1}^{n-3} g_{i+2} \vartheta_i', \\
\epsilon_{1j} & = g_{j} \vartheta_{j-3} + \sum_{i=1}^{n-3} g_{j-n+1-i}, \ (j = 4, 5, \ldots, n), \\
\epsilon_{1i} & = 0, \ (i = 2, 3, \ldots, n), \quad \epsilon_{i2} = \frac{x_{n+1-i}}{\alpha_1}, \ (i = 2, 3, \ldots, n), \\
\epsilon_{i3} & = \frac{h_2 x_{n+1-i}}{\zeta_1} + \sum_{m=1}^{n-3} h_{m+2} x_{n+1-i} \vartheta_m + \vartheta_{n-1-i}, \ (i = 2, 3, \ldots, n), \\
\epsilon_{ij} & = h_2 x_{n+1-i} \vartheta_{j-3} + \sum_{m=1}^{n-3} h_{m+2} x_{n+1-i} \vartheta_{m-3} + \vartheta_{n-1-i-j-3}, \ (i = 2, 3, \ldots, n; j = 4, 5, \ldots, n), \\
g_1 & = \frac{\alpha_2}{F_{k+n-2}}, \quad g_i = -\frac{\alpha_2 F_{k+n-i-1} - \alpha_1 F_{k+i-2}}{\alpha_1 F_{k+n-2}}, \ (i = 2, 3, \ldots, n-1), \\
h_1 & = -\frac{F_{k+n-i-1}}{\alpha_1}, \ (i = 2, 3, \ldots, n-1), \\
\zeta_1 & = -\frac{\beta_1}{\alpha_1} F_{k+n-3} + \beta_2 - V_1 \mathbb{W}_1^{-1} V_1, \quad \mathbb{V}_1 = (2F_k, F_{k+2}, 2F_{k+1}, \ldots, 2F_{k+n-6}, 2F_{k+n-5})^T, \\
\mathbb{W}_1^{-1} & = (\alpha_{i,j})_{i,j=1}^{n-3}, \quad \alpha_{i,j} = \begin{cases} \mu_{j+1}, & i \geq j, \\
0, & i < j, \end{cases} \ (i, j = 1, 2, \ldots, n-3), \\
\mu_i & = (-1)^{i-1} \lambda_i, \quad \lambda_1 = 1, \quad \lambda_2 = 2F_k, \\
\lambda_i & = 2F_k \lambda_{i-1} - F_{k-1} F_{k+2} \lambda_{i-2} + \sum_{j=2}^{i-2} (-1)^j F_k \lambda_{i-j-1}, \ (i = 3, 4, 5, \ldots, n-3), \\
\vartheta_i' & = \frac{1}{\zeta_1} \vartheta_i, \ (i = 1, 2, \ldots, n-3), \quad \vartheta_i' = -\frac{1}{\varrho_i} \vartheta_i, \ (i = 1, 2, \ldots, n-3), \\
\vartheta_i & = \sum_{j=1}^{n-3} \frac{(-1)^j}{\alpha_1} F_{k+n-3-j} + \beta_{j+2} \mu_{j+1-i}, \ (i = 1, 2, \ldots, n-3), \\
\varrho_1 & = \mu_1 \cdot 2F_{k}, \quad \varrho_2 = \mu_2 \cdot 2F_k + \mu_1 \cdot F_{k+2}, \\
\varrho_i & = \mu_i \cdot 2F_k + \mu_{i-1} \cdot F_{k+2} + \sum_{j=1}^{i-2} \mu_{i-j-1} \cdot 2F_{k+j}, \ (i = 3, 4, \ldots, n-3), \\
\alpha_{i,j} & = \mu_{i-j+1} + \frac{1}{\zeta_1} \varrho_i \vartheta_j, \ (i, j = 1, 2, \ldots, n-3), \text{ if } i - j < 1, \text{ denote } \mu_{i-j+1} = 0 \\
\alpha_1, \ \alpha_2, \ \beta_i, \ \chi_i, \ (i = 1, 2, \ldots, n-1) \text{ are the same as in Theorem 2.1.}
\end{align*}

Proof. Let $C_2$ and $D_2$ be two $n \times n$ invertible matrices, defined by

$$
C_2 = \begin{pmatrix}
1 & 0 & 1 \\
-\frac{\alpha_2}{\lambda} & 0 & 1 \\
& \ddots & \ddots & \ddots \\
& & 1 & 1 \\
& & & & \ddots
\end{pmatrix}_{n \times n}, \quad 
D_2 = \begin{pmatrix}
1 & g_1 & g_2 & \cdots & g_{n-2} & g_{n-1} \\
1 & h_2 & & & & \\
& & 1 & h_3 & \cdots & h_{n-2} \\
& & & & \ddots & h_{n-1} \\
& & & & & 1
\end{pmatrix}_{n \times n},
$$

where $g_1 = \frac{\alpha_2}{\lambda}$, $g_i = \frac{-\alpha_2 F_k + \alpha_1 F_{k+1}}{\alpha_1 F_{k+2}}$, $h_i = \frac{-F_{k+n-i-1}}{\alpha_1}$, $(i = 2, 3, \ldots, n-1)$.

Let $C_1$ and $D_1$ be as in the proof of Theorem 2.1, multiplying $C_1T_{F_k}D_1$ by $C_2$ from the left and by $D_2$ from the right, we obtain

$$
C_2C_1T_{F_k}D_1D_2 = \mathcal{N} \oplus \mathfrak{M},
$$

where $\mathcal{N} \oplus \mathfrak{M}$ is the direct sum of $\mathcal{N}$ and $\mathfrak{M}$. $\mathcal{N} = \text{diag}(-F_{k+n-2}, \alpha_1)$ is a nonsingular diagonal matrix,

$$
\mathfrak{M} = \begin{pmatrix}
-\frac{\alpha_4}{\lambda_1} F_{k+n-3} + \beta_2 & -\frac{\alpha_4}{\lambda_1} F_{k+n-4} + \beta_3 & -\frac{\alpha_4}{\lambda_1} F_{k+n-5} + \beta_4 & \cdots & -\frac{\alpha_4}{\lambda_1} F_{k+n-2} + \beta_{n-2} & -\frac{\alpha_4}{\lambda_1} F_k + \beta_{n-1} \\
F_{k+1} & 2F_k & F_{k-1} & 0 & \cdots & \\
2F_{k+1} & F_{k+2} & 2F_k & \cdots & & \\
& \ddots & \ddots & \ddots & \ddots & \cdots \\
2F_{k+n-5} & 2F_{k+n-6} & 2F_{k+n-7} & \cdots & 2F_k & F_{k-1}
\end{pmatrix}
$$

Let $\mathcal{C} = C_2C_1$ and $D = D_1D_2$, we can get,

$$
T_{F_k}^{-1} = D(\mathcal{N}^{-1} \oplus \mathfrak{M}^{-1})C,
$$

where

$$
\mathcal{C} = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & 1 \\
1 & 0 & \cdots & \cdots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & 0 & 1 & 1 & -1 \\
& & & & & \\
& & & & & \\
& & & & & \\
0 & 1 & 1 & -1 & 0 & \cdots & 0
\end{pmatrix}
$$

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Use Lemma 5 in \[\text{where}\] we obtain,

\[
\mathcal{M}^{-1} = \begin{pmatrix}
\frac{1}{\lambda_1} \mathbb{W}_1^{-1} U_1 & -\frac{1}{\lambda_1} \mathbb{W}_1^{-1} V_1 \\
-\frac{1}{\lambda_1} \mathbb{W}_1^{-1} U_1 & \mathbb{W}_1^{-1} + \frac{1}{\lambda_1} \mathbb{W}_1^{-1} U_1 \mathbb{W}_1^{-1} V_1 \mathbb{W}_1^{-1}
\end{pmatrix},
\]

where \(\mu_1, \mu_2, \ldots, \mu_n\) are the forms as in the interpretation of Theorem 2.2.
where \( \zeta_1 = -\frac{\beta_2}{\alpha_1} F_{k+n-3} + \beta_2 - V_1 W_1^{-1} U_1 \), and simply we can get \((\Omega \oplus \Omega)^{-1}\),
\[
(\Omega \oplus \Omega)^{-1} = \begin{pmatrix}
\frac{1}{F_{k+n-2}} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & \frac{1}{\Omega} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\Omega} & \delta_1 & \delta_2 & \cdots & \cdots & \delta_{n-4} & \delta_{n-3} \\
0 & 0 & \varpi_1 & \varsigma_1 & \varsigma_2 & \cdots & \cdots & \varsigma_{1,n-4} & \varsigma_{1,n-3} \\
0 & 0 & \varpi_2 & \varsigma_3 & \varsigma_4 & \cdots & \cdots & \varsigma_{2,n-4} & \varsigma_{2,n-3} \\
0 & 0 & \varpi_3 & \varsigma_5 & \varsigma_6 & \cdots & \cdots & \varsigma_{3,n-4} & \varsigma_{3,n-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \varpi_{n-4} & \varsigma_{n-4,1} & \varsigma_{n-4,2} & \cdots & \cdots & \varsigma_{n-4,n-4} & \varsigma_{n-4,n-3} \\
0 & 0 & \varpi_{n-3} & \varsigma_{n-3,1} & \varsigma_{n-3,2} & \cdots & \cdots & \varsigma_{n-3,n-4} & \varsigma_{n-3,n-3} \\
\end{pmatrix},
\]
where \( \delta_i, \varpi_i, \varsigma_{i,j} (i, j = 1, 2, \cdots, n-3) \) are the forms as in the interpretation of Theorem 2.2.

Then multiplying \((\Omega \oplus \Omega)^{-1}\) by \(D\) from the left, we can obtain
\[
D(\Omega \oplus \Omega)^{-1} = \begin{pmatrix}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13} & \cdots & \epsilon_{1,n-1} & \epsilon_{1,n} \\
\epsilon_{21} & \epsilon_{22} & \epsilon_{23} & \cdots & \epsilon_{2,n-1} & \epsilon_{2,n} \\
\epsilon_{31} & \epsilon_{32} & \epsilon_{33} & \cdots & \epsilon_{3,n-1} & \epsilon_{3,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\epsilon_{n-1,1} & \epsilon_{n-1,2} & \epsilon_{n-1,3} & \cdots & \epsilon_{n-1,n-1} & \epsilon_{n-1,n} \\
\epsilon_{n,1} & \epsilon_{n,2} & \epsilon_{n,3} & \cdots & \epsilon_{n,n-1} & \epsilon_{n,n} \\
\end{pmatrix},
\]
where \( \epsilon_{ij} (i, j = 1, 2, \cdots, n) \) are the forms as in the interpretation of Theorem 2.2.

In the end, we can obtain \(T_{F_{k,n}}^{-1}\),
\[
T_{F_{k,n}}^{-1} = D(\Omega \oplus \Omega)^{-1} C = \begin{pmatrix}
0 & \gamma_{12} & \gamma_{13} & \gamma_{14} & \cdots & \gamma_{1,n-1} & \gamma_{1n} \\
-\gamma_{12} & 0 & \gamma_{23} & \gamma_{24} & \cdots & \gamma_{2,n-1} & \gamma_{2,n} \\
-\gamma_{13} & -\gamma_{23} & 0 & \gamma_{34} & \cdots & \gamma_{3,n-1} & \gamma_{3,n} \\
-\gamma_{14} & -\gamma_{24} & -\gamma_{34} & 0 & \cdots & \gamma_{4,n-1} & \gamma_{4,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\gamma_{1,n-1} & -\gamma_{2,n-1} & -\gamma_{3,n-2} & -\gamma_{4,n-2} & \cdots & 0 & \gamma_{12} \\
-\gamma_{1,n} & -\gamma_{2,n-1} & -\gamma_{3,n-2} & -\gamma_{4,n-3} & \cdots & -\gamma_{12} & 0 \\
\end{pmatrix},
\]
where \( \gamma_{ij} (i = 1, 2, \cdots, \frac{n}{2}, j = 2, 3, \cdots, n) \) are the forms as in the interpretation of Theorem 2.2, which completes the proof.

## 3 Numerical Example

In this section, an example demonstrates the method introduced above for the calculation of determinant and inverse of the Fibonacci skew symmetric Toeplitz matrix. Here we consider a 6 \times 6 matrix, and assume \( k = 3 \), the matrix is:
\[
T_{F_{3,6}} = \begin{pmatrix}
0 & 2 & 3 & 5 & 8 & 13 \\
-2 & 0 & 2 & 3 & 5 & 8 \\
-3 & -2 & 0 & 2 & 3 & 5 \\
-5 & -3 & -2 & 0 & 2 & 3 \\
-8 & -5 & -3 & -2 & 0 & 2 \\
-13 & -8 & -5 & -3 & -2 & 0 \\
\end{pmatrix},
\]
Use the Theorem 2.1, we can get $\alpha_1 = 26 - 39i$, $\beta_1 = -3 + 5i$, and from (2.7), (2.8), (2.14), we can obtain 
$$\det G_4([\beta_i]_{i=1}^{5}, -1, -3, 0, 3, 1) = 5,$$
then from (2.1), we get
$$\det T^{-1}_{F_5, 6} = \begin{pmatrix}
0 & -\gamma_2 & \gamma_1 & \gamma_1 & \gamma_1 & \gamma_1 \\
\gamma_2 & 0 & -\gamma_2 & \gamma_1 & \gamma_1 & \gamma_1 \\
\gamma_2 & \gamma_2 & 0 & -\gamma_2 & \gamma_1 & \gamma_1 \\
\gamma_2 & \gamma_2 & \gamma_2 & 0 & -\gamma_2 & \gamma_1 \\
\gamma_2 & \gamma_2 & \gamma_2 & \gamma_2 & 0 & -\gamma_2 \\
\gamma_2 & \gamma_2 & \gamma_2 & \gamma_2 & \gamma_2 & 0
\end{pmatrix}.$$

To compute the inverse of this matrix, we can use the corresponding formulas in Theorem 2.2, and we can get, $\gamma_{12} = -\frac{2}{13}, \gamma_{13} = \frac{7}{13}, \gamma_{14} = -\frac{3}{13}, \gamma_{15} = \frac{7}{13}, \gamma_{16} = -\frac{5}{13}, \gamma_{23} = -\frac{12}{13}, \gamma_{24} = \frac{5}{13}, \gamma_{25} = -\frac{10}{13}, \gamma_{34} = -\frac{8}{13}.$
Then we can obtain,

4 Conclusion

The matrix which has the form as (1.1) has important applications in the calculation of engineering and solving least squares problems. Specially in [12], the authors gave the explicit determinants and inverses of Fibonacci skew symmetric Toeplitz matrices, which $k = 2$. And In this paper, we generalize the obtained results and by constructing the special transformation matrices we get the determinant and inverse of the skew symmetric generalized Toeplitz matrices in section 2.

Competing Interests

The author has declared that no competing interests exist.

References


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